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On Some Recent Developments in Ruin Theory and their Practical Applicability

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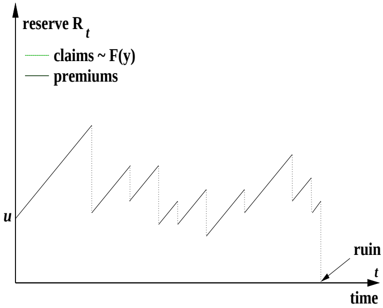
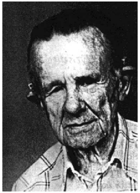
Jahrestagung SAV, ASTIN

Luzern, August 28, 2009

Non-Life Insurance Portfolio - Classical Collective Risk Model



F. Lundberg (1903)



H. Cramér

$$R_t = u + c\,t - \sum_{n=1}^{N(t)} X_n$$

Premium income: $P(t) = c\,t$

Claim payments: $X_n \ldots$ iid random variables (d.f. F)

$N(t) \ldots$ homogeneous Poisson process (λ) operational time

Initial capital u

Ruin Probability $$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 \,\middle|\, R_0 = u\right)$$

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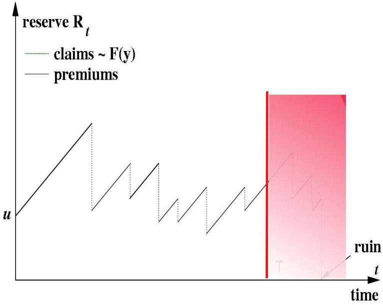

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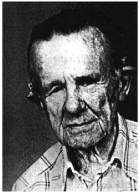

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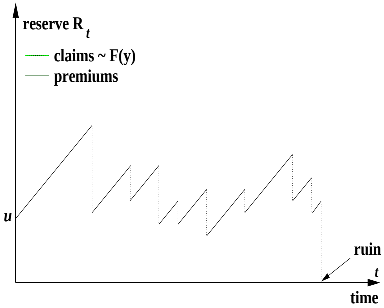
Initial capital u

Ruin Probability

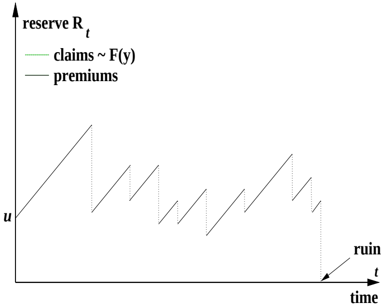
$$\boxed{\psi(u, T) = \mathbb{P}\left(\inf_{0 \le t \le T} R_t < 0 \,\middle|\, R_0 = u \right)}$$

Extensions of the Classical Model



- $N(t)$ is a renewal process (iid interclaim times)
- more general point processes
- reinsurance
- modelling the investment component:
  - inflation, interest on the surplus, debt (absolute ruin probabilities)
  - investment in financial market
  - tax payments

Extensions of the Classical Model (ctd.)



- dividend payments until ruin
- dependency
  - Markov modulation
  - Semi-Markovian behavior
  - shot-noise intensity, modulated correlation
  - adaptive premium rules,

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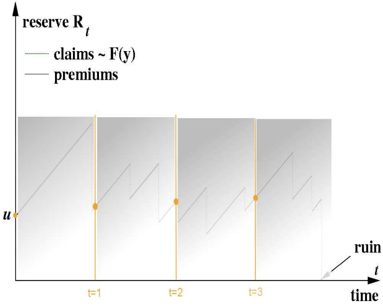
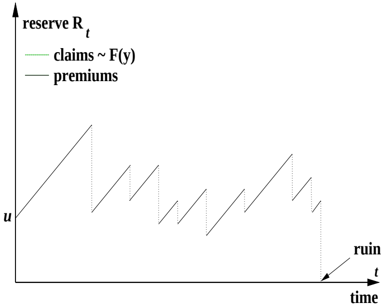


Discrete-Time Risk Model

$$R_n = u + \sum_{k=1}^{n} C_k - \sum_{k=1}^{n} X_k$$

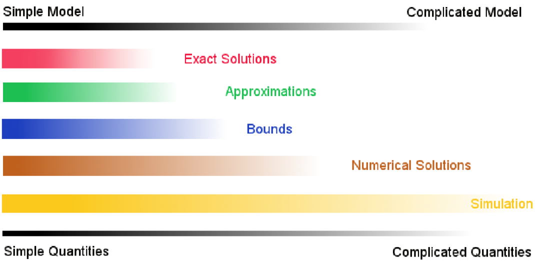
- dividend payments until ruin
- dependency
  - Markov modulation
  - Semi-Markovian behavior
  - shot-noise intensity, modulated correlation
  - adaptive premium rules, underwriting cycles, IBNR, etc.
- Incomplete information
- Combinations of all the above

Extensions of the Classical Model (ctd.)



- **Stochastic Control:** reach given objective/target!
  - Minimize ruin probability through investment
  - Minimize ruin probability through reinsurance
  - Maximize expected dividends until ruin
    - Refinements:
      Survival constraints, transaction costs, finite-time horizon,. . .
  - etc.

Trade-Off for Computability



Ruin Probability: measures the ability to fulfill obligations at all points in time (resp. in the given time interval)

Value at risk: solvability at a fixed time point

in insurance company: usually once a year

Ruin Probability: "dynamic VaR"

Long-term horizon risk management:

- no arbitrariness in time scale, stability, robustness of strategy
- long-term development
- importance for individual planning
- liquidity
- supervision, solvency discussion (e.g. 1 year, 99.5%)

Raised Disadvantages of Ruin Theory Concept:

- assumed stationarity; But: Operational Time!
- complexity of calculation; But: sometimes simple approximations available
- too few explicit formulae
- **too simple model assumptions** (e.g. claims reserving etc.)
- finite-time horizon needed; But: possible
- technical ruin is not bankruptcy; But: Liquidity Issue!

Some of above arguments may not be "final"

Important: **Way of Thinking** suggested by ruin theory

Examples:

- Need for Safety Loading
- Premium Calculation Principles (e.g. Bühlmann)
- Amount of Reinsurance (e.g. Straub)
- Applications in Finance (e.g. Geman (1998))
- Exponential Risk Measure:

$$\psi(u) \leq e^{-Ru} = \epsilon \text{ with } e^{Rc} = \mathbb{E}(e^{RS})$$

R... adjustment coefficient

$$R_\epsilon = |\log \epsilon|/u \Rightarrow \pi(S) = \rho_{\epsilon,u}(S) = c = \frac{1}{R_\epsilon} \log E(e^{R_\epsilon S})$$

(Dhaene, Goovaerts, Kaas (2003))

- VaR-type risk measure:

$$\rho_{\epsilon,c}(S) = \inf\{u \geq 0 | \psi(u) \leq \epsilon\} = \psi^{-1}(\epsilon)$$

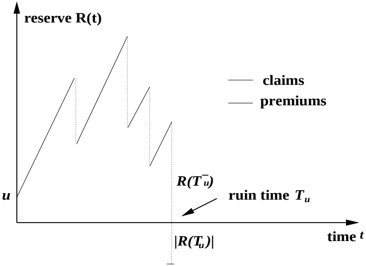
(Cheridito & Delbaen & Kupper (2006), Trufin & A. & Denuit (2009))

Multi-period risk functionals: just about to develop

Gerber-Shiu discounted penalty function

$$m_\delta(u) := \mathbb{E}\Big(w(R(T_u^-), |R(T_u)|)\, e^{-\delta T_u}\, 1_{\{T_u<\infty\}} \Big)$$

$w(x_1, x_2) \ldots$ non-negative function



Special Cases:

- $\underline{w \equiv 1}$: Laplace transform of T_u ($\delta = 0$: ruin probability).
- $\underline{w(x_1, x_2) = 1_{\{x_1 \le y\}} 1_{\{x_2 \le z\}},\ \delta = 0}$: joint distr. of $R(T_u^-)$ and $R(T_u)$.

(cf. also Segerdahl, Dickson, Dos Reis, Dufresne, Powers, Lin, Willmot, Cai, Li, Garrido, Schmidli ...)

Ruin Theory related to Mathematics

Application of

- Statistics
- Probability Theory, Stochastic Processes
  - Examples: Random Walks, Renewal Theory, Lévy Processes, Large Deviations,...
- Integral Transforms, Complex Analysis
- Asymptotic Analysis, Functional Analysis,
- Algebra, Number Theory, Symbolic Computation
- Stochastic Control Theory

Conclusion

- Ruin Theory is a vital research field
- Interaction with many branches of mathematics
- Links to other disciplines
- Bridges to Finance
- Many challenging future research issues
- "Way of Thinking" important

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- Finally.. another application of mathematics in insurance:

On the Danger of Ignoring Heavy Tails - Example: Coefficient of Variation

Reinsurance portfolio with few data points $X_1, \ldots, X_n$

typical procedure:

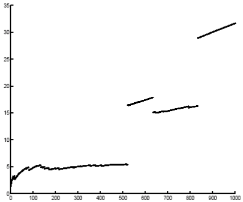
- Estimation of $\mathbb{E}(X)$ by $\hat{\mu} = (X_1 + \cdots + X_n)/n$
- Estimation of $\mathrm{CoV}(X) = \frac{std(X)}{\mathbb{E}(X)}$ from "related" portfolio
- Premium $P = \hat{\mu} + \beta\ \mathrm{CoV}(X)$

Typical claim distribution: Pareto: $P(X_i > x) = \left(1 + \frac{x}{b}\right)^{-\alpha}$ $(\alpha, b > 0)$

Industry fire: $\hat{\alpha} < 1 \longrightarrow \mathbb{E}(X) = \infty$

Houses fire: $1 < \hat{\alpha} < 2 \longrightarrow \mathbb{E}(X) < \infty$, but $Var(X) = \infty$

$\widehat{\mathrm{CoV}}(X)$ from Pareto(0.5):



A simple test for the finiteness of the mean/variance

Fuchs et al. (2001), Albrecher & Teugels (2006,2009)

$\{X_i\}_{i=1}^n$ iid pos. r.v. (d.f. F):

$$\boxed{T_n := \frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2}}$$

$$n\,T_n = \widehat{\mathrm{CoV}(X)}^2 + 1$$

$1 - F(x) \sim x^{-\alpha}\ell(x)$, $\quad \lim_{x\to\infty} \frac{\ell(tx)}{\ell(x)} = 1 \,\forall\, t > 0.$

$\alpha < 2$: DA condition

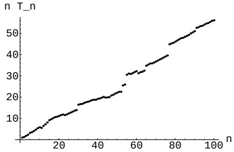
$$\beta > 0: \ \mathbb{E}(X_1^\beta) = \beta \int_0^\infty x^{\beta-1}(1 - F(x))\,dx \begin{cases} < \infty, & \beta < \alpha \\ = \infty, & \beta > \alpha \end{cases}$$

$$\boxed{T_n := \frac{X_1^2+X_2^2+\ldots+X_n^2}{(X_1+X_2+\ldots+X_n)^2}}$$

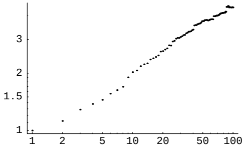
$0 < \alpha < 1$: $\lim_{n\to\infty} \mathbb{E}(T_n) = 1 - \alpha$

$1 < \alpha < 2$: $\mathbb{E}(T_n) \sim_{n\to\infty} \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mathbb{E}^2(X)}\, n^{1-\alpha}\ell(n)$

- Alternative estimator for extreme value index $1/\alpha$



Pareto ($\alpha = 0.5$)



Pareto($\alpha = 1.5$)

- Test for finiteness of $\mathbb{E}(X)$ and Var(X) (for $X \in DA(\alpha)$)