Economic Theory of Finacial Markets – 17. August 2009

- Wie kann die Risikoaversion einer Marktwirtschaft bzw. von Marktteilnehmern modelliert werden?
- Wie sieht die Von Neumann-Morgenstern räpresentation aus?
- Wie ist der indifference price for risk definiert? Interpretation?
- Welche Nutzenfunktionen kennen sie? Wie sieht die exp. Nutzenfunktion aus?
- Berechnen sie den indifference price for risk für eine Erdbebenversicherung mit Eintretenswahrscheinlichkeit p=0.1 (und exp. Nutzenfunktion)
- Leiten sie ein Mean-Variance-Portfolio her (Annahme: risikofreies Instrument existiert)
- Unter welchen Voraussetzungen existiert diese Lösung?

Economic theory of financial markets (Wüthrich):

i)

- Name the assumptions on the Markovitz problem
- Solve it, with or without riskfree rate.
- How do we use this to rate assets -> CAPM
- Compare with arbitrage on result and assumptions.

ii)

- Name the assumptions on the Markovitz problem
- Solve the Lagragian equation for the case without risk-free asset
- Plot and explain the mean-variance curves for both cases with and without risk-free asset
- What does the tangential portfolio mean?
- Assumption for APT (easier case), the pricing formula and prove.

Economical theory of financial markets

M. Wüthrich, 2011

- Utility Theory
- Mean-Variance Optimization Problem (CAPM)
- Arbitrage pricing theory (APT)
- Cash-flow valuation (stochastic discounting)

→ additional literature: Demange-Laroque, 2006, (nice)
Finance and Economics of uncertainty, Blackwell Publ.

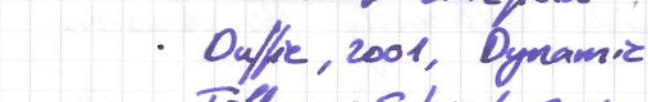
→ · Ingersall, 1987, Theory of Financial Decision making, Rowman & Littlefield, used for chapter 3.
· Duffie, 2001, Dynamic asset Pricing theory, Princeton
· Föllmer-Schied, 2004, Stochastic Finance, de Gruyter

Chapter 0: introduction

Economic Theory explains how wealth is consumed and invested.

0.1. Two-periods model

assume we have a single good that cannot be stored and it should be available at times $t=0$ and $t=1$.



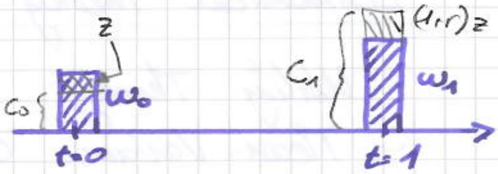
The members of the Economy can exchange this good, and decide whether they rather consume today or tomorrow.

0.1.1. Individual demand for saving

- Consider an individual financial agent with an endowment w_0 at $t=0$ and w_1 at $t=1$
- Lending and borrowing money is possible at a fixed interest rate $r > -1$

Endowment: capital, fund of capital resources invested for future needs

· Assume that this financial agent saves an amount $z \in \mathbb{R}$
($z < 0 \Rightarrow$ borrowing)
Then he has the consumption stream $\underline{c} = (c_0, c_1) \in \mathbb{R}^2$



$$(0.1) \qquad \begin{aligned} c_0 &= w_0 - z \\ c_1 &= w_1 + (1+r)z \end{aligned} \qquad \text{consumption at time } t$$

Q: Should the financial agent rather consume today or save for tomorrow.

A: Every agent will decide differently. His decision will depend on:

- $\underline{w} = (w_0, w_1)$
- interest rate r
- impatience / risk aversion

is a line

Budget Constraint

discounted to today

$$0.1 \Rightarrow \quad c_0 + \frac{c_1}{1+r} = w_0 + \frac{w_1}{1+r} \qquad (0.2)$$

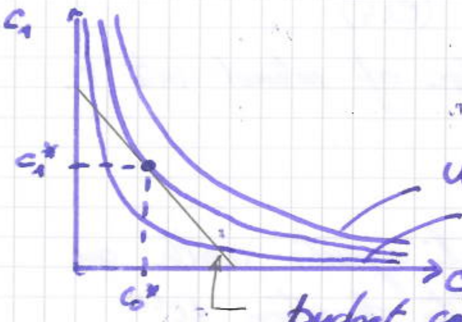
In economic theory, consumption behavior is typically modelled with utility functions

$$U: \mathbb{R}^2 \to \mathbb{R}$$
$$\underline{c} = (c_0, c_1) \mapsto U(\underline{c}) \qquad \text{"happiness of the financial agent"}$$

$$(0.4) \quad \Rightarrow \quad \operatorname{argmax}_{\substack{\underline{c} \in \mathbb{R}^2 \\ +(0.2)}} U(\underline{c}) =: \underline{c}^* \qquad \text{maximize utility, subject to (0.2)}$$

$$(0.3) \quad \text{define,} \quad U(\underline{c}) =: u(c_0) + \frac{1}{1+\delta} u(c_1), \quad u: \mathbb{R} \to \mathbb{R}, \ \delta > 0$$

economist's solution



$u_0 < u_1 < u_2$

$u_2 = U(\underline{c})$ indifference curve

$u_0 = U(\underline{c})$

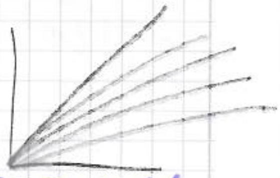
budget constraint $\Rightarrow$ • is optimal $= \underline{c}^*$

mathematical solution

$$c_1 = c_1(c_0) = (1+r)(\omega_0 + c_0) + \omega_1 \qquad \text{(budget constraint)}$$

$$\Rightarrow U(\underline{c}) = u(c_0) + (1+\delta)^{-1} u(c_1(c_0))$$

$$\frac{\partial U}{\partial c_0} \overset{!}{=} 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial c_0^2} < 0. \quad \Rightarrow c_0^* \Rightarrow c_1^*$$



Example 0.1. For $c \in \mathbb{R}_+$, $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0, \gamma \neq 1$ <!

Then, $$U(\underline{c}) = \frac{c_0^{1-\gamma}}{1-\gamma} + \frac{1}{1+\delta}\frac{c_1^{1-\gamma}}{1-\gamma}$$

$$\overset{0.2}{=} \frac{c_0^{1-\gamma}}{1-\gamma} + \frac{1}{1+\delta}\frac{(\omega_1 + (1+r)\omega_0 - (1+r)c_0)^{1-\gamma}}{1-\gamma}$$

$$\frac{\partial U}{\partial c_0} = c_0^{-\gamma} - \frac{1+r}{1+\delta}(\omega_1 + (1+r)\omega_0 - (1+r)c_0)^{-\gamma} \overset{!}{=} 0.$$

$$c_0 = \left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}(\omega_1 + (1+r)\omega_0 - (1+r)c_0)$$

$$\left[1 + (1+r)\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}\right] c_0 = \left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}(\omega_1 + (1+r)\omega_0)$$

$$c_0^* = \frac{(1+r)\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}}{1 + \left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}\cdot(1+r)} \underbrace{(\omega_0 + \omega_1/(1+r))}_{\text{total discounted endowment}}$$

$$=: \alpha(r, \delta, \gamma)\,(\omega_0 + \omega_1/(1+r))$$

$$\Rightarrow c_1^* = c_1(c_0^*) = (1-\alpha)((1+r)\omega_0 + \omega_1) \quad \text{using} \quad (1+r)(1-\alpha) = \frac{1+r}{1+(1+r)\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}}$$

$\hookrightarrow$ think about a credibility weight

now we can analyse the sensitivity of $\underline{c}^*$.

This solves the first economic problem, i.e. it describes how the financial agent maximizes (0.4)

> Note: This is subject to the knowledge of interest rate r.

0.1.2 Economic equilibrium

Assume we have N financial agents with an endowment $\omega^{(i)}$ and a fixed utility function $U^{(i)}$

Def 0.2 : An <u>equilibrium rate</u> $r > -1$ for consumption profiles $\underline{c}^{(i)} \in \mathbb{R}^2$ $(i = 1, \dots, N)$ is given iff

1) $\underline{c}^{(i)}$ maximizes $U^{(i)}$ for each agent i, subject to his budget constraint

on se prête l'un à l'autre — 2) we have <u>market clearing</u> (economic principle)

$$\sum_{i=1}^{N} c_0^{(i)} = \sum_{i=1}^{N} \omega_0^{(i)} \quad \text{(not storing apples)}$$
$$\sum_{i=1}^{N} c_1^{(i)} = \sum_{i=1}^{N} \omega_1^{(i)}$$

Remark: if such an equilibrium rate r exists
- every agent i can optimize his consumption (wrt r)
- market clearing => demand = supply

Homogeneous market: in example 0.1, if they all have $U^{(i)}(\underline{c}) = U(\underline{c})$, then hence $\gamma^{(i)} = \gamma$, $\delta^{(i)} = \delta$ and $c^{*(i)} = \operatorname{argmax} U(\underline{c})$

$$\Rightarrow \quad c_0^{*(i)} = \alpha(r, \delta, \gamma)\left(\omega_0^{(i)} + \frac{\omega_1^{(i)}}{1+r}\right)$$
$$c_1^{*(i)} = (1+r)\left(1 - \alpha(r, \delta, \gamma)\right)\left(\omega_0^{(i)} + \frac{\omega_1^{(i)}}{1+r}\right)$$

not discounted

Q: is there an equilibrium rate $r > -1$, where everyone is happy? and then, is it unique?

Market clearing:
$$\sum_{i=1}^{N} \omega_0^{(i)} = \sum_{i=1}^{N} c_0^{*(i)} = \alpha \sum_{i=1}^{N} \omega_0^{(i)} + \frac{\omega_1^{(i)}}{1+r}$$
$$\sum_{i=1}^{N} \omega_1^{(i)} = \sum_{i=1}^{N} c_1^{*(i)} = (1+r)(1-\alpha) \sum_{i=1}^{N} \omega_0^{(i)} \frac{\omega_1^{(i)}}{1+r}$$

a) $\Rightarrow (1-\alpha) \sum_{i=1}^{N} \omega_0^{(i)} = \frac{\alpha}{1+r} \sum_{i=1}^{N} \omega_1^{(i)}$ ~~(due to budget constraint~~)

γ = risk aversion, δ = impatience.

Define the growth g: $1+g = \sum_{i=1}^N w_1^{(i)} / \sum_{i=1}^N w_0^{(i)}$ (c)

$$\Rightarrow \frac{1}{1+(1+r)\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}} = (1+g)\frac{\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}}{1+(1+r)\left(\frac{1+r}{1+\delta}\right)^{-1/\gamma}}$$

$$\Rightarrow \left(\frac{1+r}{1+\delta}\right)^{1/\gamma} = 1+g \Rightarrow \underline{r^* = (1+\delta)(1+g)^{\gamma} - 1} \checkmark$$ equ. rate exists + unique

linked to growth $\square$

Aim: generalize this toy model:

1) Heterogeneous market: beliefs and utility functions of financial agents differ
2) In general, endowments are stochastic
3) In general, we have different investment possibilities, in particular, also such which have a random return.
4) Multiperiod models, discrete time model $\Rightarrow$ continuous time models.

Chapter 1: Utility theory

1.1 Expected utilities & risk aversion

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

$\mathcal{X} = \{X$ random variables on $(\Omega, \mathcal{F}, \mathbb{P})\}$

X: random payoffs of an initial investment.

Aim: introduce a preference order on $\mathcal{X}$

Def: A preference order $\succsim$ on $\mathcal{X}$ is a relation that has the following two properties:

a) completeness: $\forall X, Y \in \mathcal{X}$, we either have $X \succsim Y$ or $Y \succsim X$

b) transitivity: $\forall X, Y, Z \in \mathcal{X}$, $X \succsim Y$ and $Y \succsim Z \Rightarrow X \succsim Z$.

if $X \succsim Y$ and $Y \succsim X$, then X and Y are indifferent: $X \sim Y$.

Def: A numerical representation of $\succsim$ on $\mathcal{X}$ is a function

$$\mathcal{U}: \mathcal{X} \to \mathbb{R}, \quad X \mapsto \mathcal{U}(X) \quad \text{s.t. } \mathcal{U}(X) \geq \mathcal{U}(Y) \text{ iff } X \succsim Y.$$

Remarks:

1) A numerical representation is not unique: choose $g: \mathbb{R} \to \mathbb{R}$ strictly increasing, $V(X) = g(\mathcal{U}(X))$ gives the same $\succeq$ on $\mathcal{X}$.
2) There are necessary and sufficient conditions $(\mathcal{X}, \succeq)$ st. a numerical representation $\mathcal{U}$ for $\succeq$ on $\mathcal{X}$ exists. For details: thm 2.6 in Föllmer-Schied.

Let I be $I \subset \overline{\mathbb{R}}$ st $X \in I$ $\forall X \in \mathcal{X}$ $\mathbb{P}$-as.
eg. if $\mathcal{X} = \{X \text{ rv}, X \geq 0\}$, then $I = \overline{\mathbb{R}}_+ \cup \{0\}$.

A numerical representation $\mathcal{U}: \mathcal{X} \to \mathbb{R}$ is given as follows: choose an increasing fct $\underline{u: I \to \mathbb{R}}$ and define

(1.1) $$\mathcal{U}(X) = \mathbb{E}(u(X)) = \int_I u(x)\, dF_X(x) \quad \text{where } X \sim F_X$$

en particulier

induced preference order is $X \succeq Y$ if $X \geq Y$ as

Def 1.3. A numeric representation of the form (1.1) is called <u>von Neumann-Morgenstern</u> representation.

- Assume u is strictly increasing, then u is called <u>utility fct</u>

Interpretation in a two-period model: the financial agent has different investment possibilities at time 0 which provide $X \in \mathcal{X}$ at time 1. He will maximize his utility $\mathcal{U}(X) = \mathbb{E}(u(X))$ subject to his budget constraint.

<u>Lemma 1.4</u>

Any affine linear transformation of the utility function u provides the same preference order:

$$v(x) = \alpha + \beta u(x), \quad x \in \mathbb{R}, \beta > 0.$$

$$\forall X, Y \in \mathcal{X}, \ \mathbb{E}(v(X)) \geq \mathbb{E}(v(Y)) \iff \mathbb{E}(u(X)) \geq \mathbb{E}(u(Y)) \ \checkmark$$

proof: linearity of the expected value.

Assumption 1.5 $\mathcal{X} \subset L^1(\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow X \in \mathcal{X},\ \mathbb{E}(X) < \infty$

Def 1.6. A financial agent with utility function $u: I \to \mathbb{R}$ is called:

broad def.

- risk averse if $\forall X \in \mathcal{X}\ \ \mathbb{E}(u(X)) \leq u(\mathbb{E}(X))$
- risk neutral $=$
- risk seeking $\geq$

interpretation of risk aversion

$$X = \begin{cases} \Pi & , p = 1/2 \\ 0 & , p = 1/2 \end{cases}$$

viewed from time $t=0$, this lottery provides the value $\mathbb{E}(X) = \Pi/2$

Lemma 1.7.

Assume that the utility function is strictly concave, then we have a risk averse agent

converse is assumption.

proof: $\mathbb{E}(u(X)) \leq u(\mathbb{E}(X))$ by Jensen's Inequality $\square$

Note: if X is non-deterministic: $\mathbb{E}(u(X)) < u(\mathbb{E}(X))$

Assumption 1.8 · Assume that the utility fcts are three differentiable.

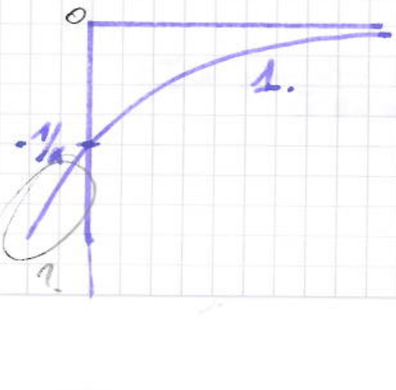
· Risk averse agents have a strictly concave utility fct.

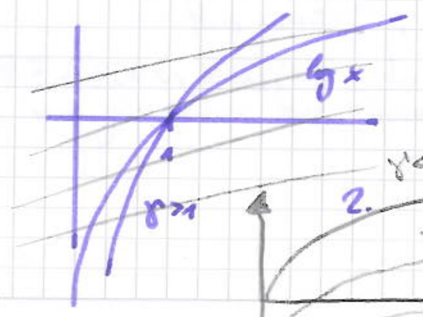
réciproque de lem 1.7

$\Rightarrow u' > 0,\ \ u'' < 0$ on I.

Examples

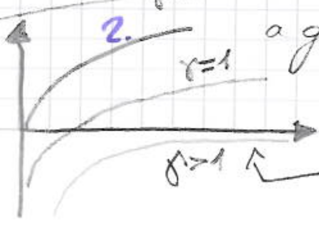
1. Exponential utility fct. $I = \mathbb{R}$, $\alpha > 0$, $u(x) = -\frac{1}{\alpha} e^{-\alpha x}$
   $u'(x) = e^{-\alpha x} > 0,\ \ u''(x) = -\alpha e^{-\alpha x} < 0$
2. Power utility fct. $I = \mathbb{R}_+$, $\gamma > 0$ $\ u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\ \log(x), & \gamma = 1 \end{cases}$
   $u'(x) = x^{-\gamma} > 0,\ \ u''(x) = -\gamma x^{-\gamma-1} < 0$





Both risk averse.

Economists believe that a good choice of $\gamma \in [2, 5]$



plus de courbure puisque borné par 0

Michael Phillipp

Risk aversion :

$$\rho_{ARA}(x) = -\frac{u''(x)}{u'(x)}$$ absolute risk aversion, Arrow-Pratt risk aversion

$$\rho_{RRA}(x) = -x\frac{u''(x)}{u'(x)}$$ relative risk aversion

exponential utility: $\rho_{ARA}(x) = \alpha > 0$, $\rho_{RRA}(x) = x\alpha$
power utility: $\rho_{ARA}(x) = \gamma/x > 0$, $\rho_{RRA}(x) = \gamma > 0$. ✓

1.2. Indifference pricing

Assume that a financial agent has an initial wealth $w \in \mathbb{R}$ and a given utility function u.

Def 1.9. Choose $X \in \mathcal{X}$. A certainty equivalent x for X is the solution to $$u(w+x) = \mathbb{E}\big(u(w+X)\big)$$

Remarks: So we say that $w+x \sim w+X \Rightarrow x \sim X$
- of course $w+x$ and $w+X$ need to be in the domain of u.
- $x = x(X, w, u)$ "deterministic price" for risky position X.

Lemma 1.10.
Assume u is a risk averse utility function then $x \le \mathbb{E}(X)$

$X = \begin{cases} M & p=1/2 \\ 0 & p=1/2 \end{cases}$
$\Rightarrow x \le M/2$

proof: $u(w+x) = \mathbb{E}(u(w+X)) \le_{\text{Jensen}} u(\mathbb{E}(w+X)) = u(w+\mathbb{E}(X))$
u strictly increasing, so $x \le \mathbb{E}(X)$ □

note: if X is non-deterministic, then $x < \mathbb{E}(X)$ ✓

All assumptions: 1) there is a preference order of von Neumann-Morgenstern.
2) u is strictly increasing, 3-times diff.
3) there is $I \subset \overline{\mathbb{R}}$ interval $\mathcal{X} = \{$set of rv on $(\Omega, \mathcal{F}, \mathbb{P})$ with image in $I\}$
$\rightarrow \mathcal{X} \subset \mathcal{M}(I, \mathcal{B})$ convex and containing pure points δ_x

Definition 1.11 Agent 1 with utility fct u_1 is <u>more risk averse</u> than agent 2 with u_2 if

$$u_1^{-1}(\mathbb{E}(u_1(X))) \le u_2^{-1}(\mathbb{E}(u_2(X))) \quad \forall X \in \mathcal{X}.$$

$X \in \mathcal{X} = \{X \text{ r.v. on } (\Omega, \mathcal{F}, \mathbb{P}) \text{ with image in } I \subseteq \overline{\mathbb{R}}\}$ interval

$\Rightarrow$ probability distribution $\mu_X(\cdot) = \mathbb{P}(X \in \cdot)$ on $(I, \mathcal{B})$

$\Rightarrow$ $\mathcal{X}$ can also be understood as a subset in

is a Π $\rightarrow$ $\mathcal{M}_1(I, \mathcal{B})$, set of all prob. distributions on $(I, \mathcal{B})$

Assumption 1.12 $\mathcal{X} \subset \mathcal{M}_1(I, \mathcal{B})$ is convex and $\mathcal{X}$ contains all pure point measures δ_x, $x \in I$. deterministic positions

! <u>Prop. 1.13.</u>

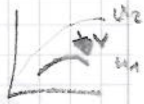
Under 1.12, the following 3 statements are equivalent:

(1) agent 1 is more risk averse than agent 2

(2) $\rho_{ARA}^{(1)}(y) \ge \rho_{ARA}^{(2)}(y) \quad \forall y \in I$.

(3) $\exists\, v$ strictly increasing and concave st.

$u_1(y) = v(u_2(y)) \quad \forall y \in I$. "more risk averse" deformation of u_2



proof: $v(x) = u_1(u_2^{-1}(x))$ is well defined. because $u_i : \underset{\text{same } I}{I} \to \mathbb{R}$

$v(u_2(y)) = u_1(y) \quad y \in I$, $x \in \mathbb{R}$ here

set $z = z(x) = u_2^{-1}(x) \in I$

$$\frac{dz}{dx}(x) = \frac{1}{u_2'(z)} \Rightarrow v'(x) = u_1'(z)(u_2^{-1})'(x) = \frac{u_1'(z)^{>0}}{u_2'(z)_{>0}} > 0 \quad \text{str. inc.}$$

$$v''(x) = \frac{u_1''(z)}{u_2'(z)} \cdot \frac{1}{u_2'(z)} - \frac{u_1'(z)}{(u_2'(z))^2} \cdot u_2''(z) \frac{1}{u_2'(z)}$$

$$= \frac{u_1'(z)}{(u_2'(z))^2} \cdot \left[\frac{u_1''(z)}{u_1'(z)} - \frac{u_2''(z)}{u_2'(z)} \right]$$

$$= \underset{>0}{\frac{u_1'(z)}{u_2'(z)^2}} \Big[\underbrace{\rho_{ARA}^{(2)}(z) - \rho_{ARA}^{(1)}(z)}_{(3)\ \le 0 \ \Leftrightarrow\ (2)} \Big]$$

$x = f \circ f^{-1}(x) \Rightarrow 1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$

$\Rightarrow (f^{-1})'(x) = 1 / f'(f^{-1}(x))$

$z = u_2^{-1}(x) \Rightarrow z'(x) = 1 / u_2'(u_2^{-1}(x))$

$= 1 / u_2'(z)$

$$(1) \iff u_1^{-1}(\mathbb{E}(u_1(X))) \le u_2^{-1}(\mathbb{E}(u_2(X))) \quad \forall X \in \mathcal{X}$$
$$\iff \mathbb{E}(u_1(X)) \le u_1(u_2^{-1}(\mathbb{E}(u_2(X)))) \quad \forall X \in \mathcal{X}$$
$$\iff \mathbb{E}(u_1(X)) \le v(\mathbb{E}(u_2(X))) \quad \forall X \in \mathcal{X}$$

$(3) \Rightarrow (1)$ $$v(\mathbb{E}(u_2(X))) \overset{(3)}{\underset{J.}{\ge}} \mathbb{E}(v(u_2(X))) \quad \text{concave}$$
$$= \mathbb{E}(u_1(X)) \quad \forall X \in \mathcal{X}.$$

$\neg(2) \Rightarrow \neg(1)$ $\exists\, \mathcal{O} \subset I$ non-empty ~~open~~ set because all nice

st. $f_{ARA}^{(2)}(z) > f_{ARA}^{(1)}(z) \quad \forall z \in \mathcal{O}$ neighborhood.

$\Rightarrow \exists\, v$ strictly increasing and convex on $\mathcal{O}$

Choose a measure $\mu \in \mathcal{X} \subset \mathcal{M}_1(I, \mathcal{B})$ that is supported in $\mathcal{O}$ and which is not concentrated in a single point $(\exists\, z_1 \ne z_2 \in \mathcal{O},\ \delta_{z_1}, \delta_{z_2} \in \mathcal{X},\ \frac{1}{2}\delta_{z_1} + \frac{1}{2}\delta_{z_2} \in \mathcal{X})$

$\neg(2) \Rightarrow$ v is strict convex

Denote $Y \sim \mu$

$$\mathbb{E}(u_1(Y)) = \mathbb{E}(v(u_2(Y))) > v(\mathbb{E}(u_2(Y)))$$
$$= u_1(u_2^{-1}(\mathbb{E}(u_2(Y))))$$
$$\Rightarrow u_1(\mathbb{E}(u_1(Y))) > u_2^{-1}(\mathbb{E}(u_2(Y))) \Rightarrow \neg(1).$$

$\square$

<u>Corollary 1.14.</u> $f_{ARA}^{(1)}(y) \ge f_{ARA}^{(2)}(y) \quad \forall y \in I$

$$\iff x^{(1)} = x(X, \omega, u^{(1)}) \le x^{(2)} = x(X, \omega, u^{(2)}) \quad \forall X \in \mathcal{X}$$

proof: $f_{ARA}^{(1)}(y) \ge f_{ARA}^{(2)}(y) \quad \forall y \in I$

$$\iff u_1^{-1}(\mathbb{E}(u_1(\omega + X))) \le u_2^{-1}(\mathbb{E}(u_2(\omega + X)))$$
$$\iff u_1^{-1}(u_1(x^{(1)})) \le u_2^{-1}(u_2(x^{(2)}))$$
$$\iff x^{(1)} \le x^{(2)}$$

Def 1.15 Assume that the financial agent has initial wealth w and utility function u. The <u>indifference price</u> $\pi = \pi(X, \omega, u)$ for $X \in \mathcal{X}$ is the solution to

$$u(\omega) = \mathbb{E}(u(w + \pi - X)) \quad \Rightarrow \omega \sim \omega + \pi - X$$

$\pi_r = \pi - \mathbb{E}(X)$ risk loading.

Lemma 1.16

u risk averse $\Rightarrow \Pi_r \geq 0$

proof (as exercise) immediate with Jensen.

Theorem 1.17

Assume u risk averse utility function.
The following two statements are equivalent.

(1) Π_r does not depend on ω

(2) $u(x) = -A \exp\{-\alpha x\} + B$ exponential $\alpha, A > 0$, $B \in \mathbb{R}$

proof: ($\Leftarrow$) $\cancel{A} \exp\{-\alpha\omega\} + \cancel{B} = u(\omega)$

$$= \mathbb{E}\left(A \exp\{-\alpha(\omega + \Pi - X)\} + B\right)$$

$$\Rightarrow 1 = \exp(-\alpha\Pi)\, \mathbb{E}(\exp(\alpha X))$$

$$\Rightarrow \Pi = \frac{1}{\alpha} \log \mathbb{E}(\exp\{\alpha X\}) \Rightarrow (1)$$

($\Rightarrow$) $\Pi = \Pi(X, u)$ not depending on ω.

$$\Rightarrow \frac{\partial \Pi}{d\omega} = 0.$$

$$u'(\omega) = \mathbb{E}\left(u'(\omega + \Pi - X)(1 + \Pi'(\omega))\right) \qquad (1.2)$$

$$= \mathbb{E}(u'(\omega + \Pi - X))$$

and $u(\omega) = \mathbb{E}(u(\omega + \Pi - X))$

By assumption, $u' > 0$ and $u'' < 0$.

def: $v(x) = -u'(x) \Rightarrow v'(x) = -u''(x) > 0 \Rightarrow$ utility fct.

$\Rightarrow V(\omega) = \mathbb{E}(v(\omega + \Pi - X))$ Π also indiff. price.

$\Rightarrow \omega = u^{-1}(\mathbb{E}(u(\omega + \Pi - X))) = v^{-1}(\mathbb{E}(v(\omega + \Pi - X)))$ $\forall X, \omega$.

$\Rightarrow$ 1.13. Agents u and v have the same risk aversion

$\Rightarrow$ 1.13. $\rho^u_{ARA}(x) = \rho^v_{ARA}(x)$ $\forall x \in \mathbb{R}$. ✓

$$\Rightarrow -\frac{u''}{u'} = -\frac{v''}{v'} = -\frac{u'''}{u''}$$

$$\Rightarrow \frac{d}{dx} \rho_{ARA}^{(u)}(x) = -\frac{u'''(x)}{u'(x)} + \frac{u''(x)^2}{u'(x)^2}$$

$$= -\frac{u''(x)}{u'(x)} \Big[\underbrace{\frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)}}_{=0} \Big] = 0.$$

$$\Rightarrow \rho_{ARA}^{(u)}(x) = -\frac{u''(x)}{u'(x)} = +A$$

$$\Rightarrow \quad A u'(x) + u''(x) = 0 \quad \Rightarrow u(x) = -A \exp\{-\alpha x\} + B. \checkmark$$

$\square$

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Indifference price $\pi = \pi(X, w, u)$

X risky position, w initial capital, u utility f.

$u(w) \overset{!}{=} \mathbb{E}(u(w + \pi - X))$

In general, economists say that $\pi(w) \searrow$ in w : ~~risk averse utility~~

Proposition 1.18.

u risk averse utility function

The following two are equivalent

1) $\pi(w) \searrow w \quad \forall X \in \mathcal{X}$

2) $\rho_{ARA}(x) = -u''(x)/u'(x)$ is decreasing in x.

Power utility f.: $\rho_{ARA}(x) = \alpha x^{-1}, \ x > 0$

Select any $X \in \mathcal{X}$, then ...

proof: 1) $\Leftrightarrow \pi'(w) \leq 0 \quad \forall X \in \mathcal{X}$

$$\Leftrightarrow u'(w) = \mathbb{E}\big(u'(w + \pi - X)(1 + \underset{\leq 0}{\pi'(w)})\big)$$

$$\leq \mathbb{E}(u'(w + \pi - X)) \quad \forall X \text{ with associated } \pi_X$$

$$\Leftrightarrow -u'(w) \geq \mathbb{E}(-u'(w + \pi - X)) \quad \forall X \qquad v = -u' \text{ also utility f.}$$

$$\Leftrightarrow v(w) \geq \mathbb{E}(v(w + \pi - X)) \quad \forall X \longrightarrow \pi \text{ is not sufficient to be indiff price}$$

$$\overset{1.13}{\underset{1.14}{\Longleftrightarrow}} \quad \rho_{ARA}^{v}(x) \geq \rho_{ARA}^{u}(x) \quad \forall x \in I$$

$$\Leftrightarrow -\frac{u'''(x)}{u''(x)} = -\frac{v''(x)}{-v'(x)} \geq -\frac{u''(x)}{u'(x)} \quad \forall x \in I$$

$$\Leftrightarrow \frac{d}{dx} \rho_{ARA}^{u}(x) = \underset{>0}{-\frac{u''(x)}{u'(x)}} \Big[\frac{u'''(x)}{u''(x)} \underset{\leq 0}{-} \frac{u''(x)}{u'(x)} \Big] \leq 0$$

$\Leftrightarrow$ 2).

$\square$

1.3. Exchange economy

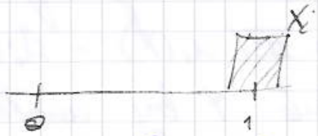
$(\Omega, \mathcal{F}, \mathbb{P})$ finite probability space $|\Omega| < \infty$

$\mathcal{X} = \{$all strict positive rv X on $(\Omega, \mathcal{F}, \mathbb{P})$

We have N financial agents with an initial risky position $X_i \in \mathcal{X}$, $i = 1, \dots, N$.

Market capitalisation

$Z = \sum_{i=1}^{N} X_i$



Assume that we have a financial agent pricing functional $\varphi \in \mathcal{X}$ with $\underline{\varphi > 0}$ $\mathbb{P}$-as. $\mathbb{E}(\varphi) = 1$ ← for risk free such that $\pi(X) = \mathbb{E}(\varphi X)$ $\forall X \in \mathcal{X}$, price for X at time 0.

φ is called
- financial pricing kernel
- state price density
- stochastic discount
- state price deflator

Aim: construct φ using the utility maximization of each agent i, and a market equilibrium condition.

Assume each agent has a given utility function u_i, $i = 1, \dots, N$. He holds initial position X_i at time 0 (before trading) and by trading he is able to achieve any position $Y_i \in \mathcal{X}$, subject to his budget constraint

$$Y_i = \underset{Y \in \mathcal{X}}{\arg\max}\; \mathbb{E}(u_i(Y)) \qquad (1.4)$$

with $\pi(X_i) = \pi(Y_i)$

π is the official market price. We don't adjust the price for each agent, we only set the "market" price.

Theorem 1.19. (First order conditions)

The optimum asset allocation $Y_i \in \mathcal{X}$ of agent i is given by $u_i'(Y_i) = \Lambda_i \varphi$ for some $\Lambda_i > 0$.

$$\Rightarrow \quad Y_i = (u_i')^{-1}(\Lambda_i \varphi)$$

given there exist this φ s.t. $\pi(X) = \mathbb{E}(\varphi X)$.

proof: method of Lagrange.

$$\mathcal{L} = \mathbb{E}(u_i(Y_i)) - \Lambda_i(\pi(Y_i) - \pi(X_i))$$

Perturb Y by another portfolio $\tilde{Y}$ and $\varepsilon \in \mathbb{R}$ small

$Y + \varepsilon \tilde{Y} > 0$ $\mathbb{P}$-a.s. → sinon comment faire varier Y?

$$\mathcal{L}(\varepsilon) = \mathbb{E}(u_i(Y + \varepsilon\tilde{Y})) - \Lambda_i(\pi(Y + \varepsilon\tilde{Y}) - \pi(X))$$

$$\frac{\partial \mathcal{L}(\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \overset{!}{=} 0 \quad \forall \tilde{Y} \in \mathcal{X} \Rightarrow \text{solution to 1.4.}$$

$$\mathcal{L}'(\varepsilon)\big|_{\varepsilon=0} \overset{\pi(X)=\mathbb{E}(\varphi X)}{=} \mathbb{E}(u_i'(Y + \varepsilon Y)\tilde{Y}) - \Lambda_i \mathbb{E}(\varphi\tilde{Y})\big|_{\varepsilon=0}$$

$\mathbb{E}(XZ) = \mathbb{E}(YZ)$ $\forall Z \in \mathcal{X}$ $\Rightarrow X = Y$

$$= \mathbb{E}(u_i'(Y)\tilde{Y}) - \Lambda\mathbb{E}(\varphi\tilde{Y}) = 0$$

$$\Rightarrow \mathbb{E}(u_i'(Y)\tilde{Y}) = \Lambda_i \mathbb{E}(\varphi\tilde{Y}) \quad \underline{\forall \tilde{Y} \in \mathcal{X} \quad (\forall \mathcal{F}\text{-meas. } \tilde{Y} \in \mathcal{X})}$$

$$\Rightarrow u_i'(Y) = \mathbb{E}(\Lambda_i\varphi \mid \mathcal{F}) = \Lambda_i \mathbb{E}(\varphi \mid \mathcal{F}) = \Lambda_i\varphi \quad \mathbb{P}\text{-a.s.}$$

$$\underbrace{u_i'(Y_i)}_{>0} = \Lambda_i \underbrace{\varphi}_{>0} \Rightarrow \Lambda_i > 0$$

✓ □

Corollary 1.20.

All optimal asset allocation Y_i are comonotonic. (Herd instinct)

proof: $Y_i = \underbrace{(u_i')^{-1}}_{<0}(\underbrace{\Lambda_i\varphi}_{>0})$ decreasing function of φ □.

Def: $\varphi, Y_1, \dots, Y_N$ <u>comonotonic</u>, if $\exists f_1, \dots, f_N \nearrow$ (or $\searrow$) such that $Y_i = f_i(\varphi)$

Assumption 1.21 (Economic Principle)

Assume we have a risk exchange economy

market clearing $$Z = \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} Y_i \qquad (1.6)$$

Theorem 1.22.

Under (1.6), Y_i the optimal asset allocations $i=1,..,N$ and Z are comonotonic.

proof: $Z = \sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} (u_i')^{-1}(\lambda_i \varphi)$ $\square$

Interpretation

Aggregate market risk Z, and then all optimal (k) asset allocations Y_i are comonotonic with Z: individual risks are eliminated and there is only an aggregate market risk left.

Example 1.23 (exponential utility)

$u_i(x) = -\frac{1}{\kappa_i} e^{-\kappa_i x} \qquad \kappa_i > 0$

$u_i'(x) = e^{-\kappa_i x}$

$(u_i')^{-1}(y) = -\frac{1}{\kappa_i} \log y$

$\Rightarrow$ thm 1.19. $Y_i = -\frac{1}{\kappa_i} \log(\lambda_i \varphi) = -\frac{1}{\kappa_i} \log \lambda_i - \frac{1}{\kappa_i} \log(\varphi)$.

$\Rightarrow$ market clearing $Z = \sum_{i=1}^{N} X_i$

$= \sum_{i=1}^{N} -\frac{1}{\kappa_i} \log(\lambda_i) - \frac{1}{\kappa_i} \log(\varphi)$

$= -\log \varphi \cdot \underbrace{\sum_{i=1}^{N} 1/\kappa_i}_{\kappa^{-1}} - \sum_{i=1}^{N} \log \lambda_i / \kappa_i$

$\kappa = (\sum 1/\kappa_i)^{-1} > 0$ aggregate risk aversion

$K := +\alpha \sum_{i=1}^{N} \log \lambda_i / \kappa_i$ weighted average of the $\log(\lambda_i)$ by $1/\kappa_i$

$\Rightarrow Z = -\kappa^{-1} \log \varphi - \kappa^{-1} K$

$\Rightarrow \underline{\varphi = e^{-K-\kappa Z}}$

$1 = \mathbb{E}(\varphi) = e^{-K}\, \mathbb{E}(e^{-\kappa Z}) \Rightarrow K = -\log \mathbb{E}(e^{-\kappa Z})$

$$\Rightarrow \underline{\varphi = \frac{e^{-\kappa Z}}{\mathbb{E}(e^{-\kappa Z})}}$$

equilibrium deflator such that everyone can achieve Y_i and market clearing.

$$\Pi(X) = \mathbb{E}(\varphi X) = \frac{\mathbb{E}(X \cdot e^{-\kappa Z})}{\mathbb{E}(e^{-\kappa Z})}$$

Esscher premium calculation principle by Bühlmann, 1980.

Example 1.23

$(\Omega, \mathcal{F}, P)$ finite proba space

$N \geq 2$ financial agents each holding an initial position $x_i > 0$ and having $u_i = -\frac{1}{\alpha_i}\exp\{-\alpha_i x\}$, $\alpha_i > 0$

Under the market clearing condition $Z = \sum_i X_i = \sum_i Y_i$

deflator
$$\Rightarrow \varphi = \frac{e^{-\alpha Z}}{\mathbb{E}(e^{-\alpha Z})}, \quad \alpha = \left(\sum_i 1/\alpha_i\right)^{-1} \text{ agg. market risk aversion}$$

$$\Rightarrow \pi(X) = \mathbb{E}(\varphi X)$$

$$Y_i = -\frac{1}{\alpha_i}\log \lambda_i - \frac{1}{\alpha_i}\log\varphi \quad , \quad \underset{\text{budget constraint}}{\pi(X_i) = \pi(Y_i)}$$
$$= -\frac{1}{\alpha_i}\log\lambda_i + \frac{1}{\alpha_i}\alpha Z - \frac{1}{\alpha_i}\log\frac{1}{\mathbb{E}(e^{-\alpha Z})}$$
$$= \frac{\alpha}{\alpha_i} Z - \frac{1}{\alpha_i}\log\frac{\lambda_i}{\mathbb{E}[e^{-\alpha Z}]}$$

$$\pi(X_i) = \pi(Y_i) = \mathbb{E}(\varphi Y_i) = \frac{\alpha}{\alpha_i}\mathbb{E}(\varphi Z) - \frac{1}{\alpha_i}\log\frac{\lambda_i}{\mathbb{E}(e^{-\alpha Z})}\underbrace{\mathbb{E}(\varphi)}_{=1}$$
$$= \frac{\alpha}{\alpha_i}\pi(Z) - \frac{1}{\alpha_i}\log\frac{\lambda_i}{\mathbb{E}(e^{-\alpha Z})}$$

$$\Rightarrow -\frac{1}{\alpha_i}\log\frac{\lambda_i}{\mathbb{E}(e^{-\alpha Z})} = \pi(X_i) - \frac{\alpha}{\alpha_i}\pi(Z)$$

$$Y_i = \frac{\alpha}{\alpha_i}(Z - \pi(Z)) + \pi(X_i)$$
$$= \underbrace{\frac{\alpha}{\alpha_i} Z}_{\text{risky position}} + \underbrace{\pi(X_i) - \frac{\alpha}{\alpha_i}\pi(Z)}_{\text{riskless position}}$$

$\Rightarrow$ all financial agents hold a linear combination of the riskless portfolio 1 and the market portfolio Z.

$\Rightarrow$ if i is more risk averse than j ($\alpha_i \geq \alpha_j$), then $\frac{\alpha}{\alpha_i} < \frac{\alpha}{\alpha_j}$, i will invest less in the risky portfolio Z.

$$\mathbb{E}(Y_i) = \frac{\alpha}{\alpha_i}\left(\mathbb{E}(Z) - \pi(Z)\right) + \pi(X_i)$$

Lemma: $\quad \mathbb{E}(Z) \geq \pi(Z)$

$\Rightarrow$ less risk averse agent has a higher expected value (has 1.)

proof: $\pi(z) = \mathbb{E}(\varphi z) = \dfrac{\mathbb{E}(ze^{-\alpha z})}{\mathbb{E}(e^{-\alpha z})} = \dfrac{d}{dr} - \log \mathbb{E}(e^{-rz})\Big|_{r=\alpha}$

$\mathbb{E}(z) = \dfrac{d}{dr} \cdot \log \mathbb{E}\left(e^{-rz}\right)\Big|_{r=0}$

claim: $\dfrac{d^2}{dr^2} \log \mathbb{E}(e^{-rz}) \leq 0$

$\Rightarrow \mathbb{E}(z) = \dfrac{d}{dr} - \log \mathbb{E}(e^{-rz})\Big|_{r=0} \geq \dfrac{d}{dr} - \log \mathbb{E}(e^{-rz})\Big|_{r=\alpha} = \pi(z)$

$\dfrac{d}{dr} \dfrac{\mathbb{E}(ze^{-rz})}{\mathbb{E}(e^{-rz})} = \dfrac{\mathbb{E}(z^2 e^{-rz}) \cdot \mathbb{E}(e^{-rz}) - \mathbb{E}(ze^{-rz})}{\mathbb{E}(e^{-rz})^2}$

$= -\left(\dfrac{\mathbb{E}(z^2 e^{-rz})}{\mathbb{E}(e^{-rz})} - \left(\dfrac{\mathbb{E}(ze^{-rz})}{\mathbb{E}(e^{-rz})}\right)^2 \right) \longrightarrow$ Variance of a Proba distortion of z

Assume $z \sim F$

Escher transform: then $dF_r(x) = \dfrac{e^{rz}}{\mathbb{E}(e^{-rz})}\, dF(x) \sim \tilde{z}_r$

non deterministic: strict inequ.

$= -\left(\mathbb{E}(\tilde{z}_r^2) - \mathbb{E}(\tilde{z}_r)^2 \right) = -\mathrm{Var}(\tilde{z}_r) \leq 0$ $\square$

Remark: This exponential utility function example is almost the only that provides a closed form solution.

Chapter 2 : Mean-Variance Analysis

2.1. Financial markets and returns

Assume we have $n+1$ financial assets with initial values $S_i^{(0)}$ at time 0 and values $S_i^{(1)}$ at time 1, $i=0,\dots,n$

Of course, if a financial deflator φ is given:

$$S_i^{(0)} = \mathbb{E}(\varphi S_i^{(1)}) \quad \text{(no arbitrage condition)}$$

$S_0^{(0)}, S_0^{(1)}$ risk-free asset ($S_0^{(1)}$ is deterministic seen from 0)

$S_i^{(0)}, S_i^{(1)}$ risky assets

An asset allocation (portfolio)

$$\tilde{\underline{a}} = (a_0, \dots, a_n) \in \mathbb{R}^{n+1}$$

$$\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}^n \quad \text{(we drop the risk-free)}$$

$\tilde{\underline{a}}$ has initial value at time 0 : $w_0 = W_0(\tilde{\underline{a}}) = \sum_{i=0}^n a_i S_i^{(0)}$ and it generates value at time 1

$$
\begin{aligned}
W_1 = W_1(\tilde{\underline{a}}) &= \sum_{i=0}^n a_i S_i^{(1)} - w_0 + w_0 \\
&= w_0 + \sum_{i=0}^n a_i (S_i^{(1)} - S_i^{(0)}) \\
&= w_0 \left(1 + \sum_{i=0}^n \frac{a_i S_i^{(0)}}{w_0} \, \frac{S_i^{(1)} - S_i^{(0)}}{S_i^{(0)}}\right) \\
&= w_0 \left(1 + \sum_{i=0}^n x_i R_i\right)
\end{aligned}
$$

$x_i = \frac{a_i S_i^{(0)}}{w_0}$, $\sum x_i = 1$

$R_i = \frac{S_i^{(1)} - S_i^{(0)}}{S_i^{(0)}}$ (random) return of asset i.

$$\Rightarrow W_1(\tilde{\underline{a}}) = w_0 (1 + \tilde{\underline{x}}' \tilde{\underline{R}}) = W_1(\tilde{\underline{x}})$$

$\Rightarrow$ either study $W_1(\tilde{\underline{a}})$: asset allocation
or $W_1(\tilde{\underline{x}})$: portfolio weights

X: normalized

with $\tilde{\underline{e}} := (1, \dots, 1)' \in \mathbb{R}^{n+1}$, $\tilde{\underline{e}}'\tilde{\underline{x}} = \sum x_i = 1$

W: ___⌐‾‾‾⌐___ ___⌐‾‾‾⌐___

Portfolio optimization under utility function u means:

$$\underline{\tilde{a}}^* = \underset{\substack{\underline{\tilde{a}} \in \mathbb{R}^{n+1}, \\ \sum a_i S_i^{(0)} = w_0, \\ \mathbb{E}(w_1) = w_1}}{\operatorname{argmax}} \mathbb{E}(u(w_1(\underline{\tilde{a}})))$$

later at last w_1 ← right price?

$$\Leftrightarrow \quad \underline{\tilde{x}}^* = \underset{\substack{\underline{\tilde{x}} \in \mathbb{R}^{n+1}, \\ \tilde{\varepsilon}' \underline{\tilde{x}} = 1, \ \tilde{x}^T \mathbb{E}(\tilde{R}) = r}}{\operatorname{argmax}} \mathbb{E}(u(w_1(\underline{\tilde{x}}))) \qquad (2.1)$$

$$w_1 = \mathbb{E}(w_1(\tilde{\underline{a}})) = w_0(1 + \underline{\tilde{x}}' \mathbb{E}(\underline{\tilde{R}}))$$
$$\Leftrightarrow \underline{\tilde{x}}' \mathbb{E}(\underline{\tilde{R}}) = \frac{w_1}{w_0} - 1 := r$$

2.2. Motivation of the mean-variance framework

1. In general, (2.1) can take a quite complex form, therefore approximations are used. (Taylor Series)
2. Here, we give an explicit model that provides the Markowitz framework.

Assume that $\cdot\ u(x) = -\frac{1}{\alpha}\exp(-\alpha x)$, $\alpha > 0$. (multivariate $\mathcal{N}$ only for motivation)

Assumption 2.1. $\cdot\ \underline{R} \sim \mathcal{N}(\underline{\mu}, \Sigma)$ (no term 0, deterministic)

$\underline{\mu} = (\mu_1, \dots, \mu_n)' = (\mathbb{E}(R_1), \dots, \mathbb{E}(R_n))' \in \mathbb{R}^n$

Σ positive definite covariance matrix of R.

$\underline{\tilde{\mu}} = (\mu_0, \underline{\mu})$, with $\mu_0 = \mathbb{E}(R_0) = R_0$ ✓

Lemma: Under assumptions 2.1.,

$$\mathbb{E}(u(W_1)) = -\frac{1}{\alpha}\exp\left\{-\alpha w_0(1 + \underline{\tilde{x}}'\underline{\tilde{\mu}}) + \frac{\alpha^2 w_0^2}{2}\underline{x}'\Sigma\underline{x}\right\}$$

proof: $W_1 = w_0(1 + \underline{\tilde{x}}'\underline{\tilde{R}}) = w_0(1 + x_0\mu_0 + \underline{x}'\underline{R})$

$\underline{x}'\underline{R} \sim \mathcal{N}(\underline{x}'\underline{\mu}, \underline{x}'\Sigma\underline{x})$

$\Rightarrow w_0(1 + x_0\mu_0 + \underline{x}'\underline{R}) \sim \mathcal{N}(w_0(1 + \underline{\tilde{x}}'\underline{\tilde{\mu}}), w_0^2\underline{x}'\Sigma\underline{x})$

$\Rightarrow \mathbb{E}(u(w_1)) = \mathbb{E}(-\frac{1}{\alpha}\exp\{-\alpha W_1\})$ ← log $\mathcal{N}$ distrib.

$= -\frac{1}{\alpha}\exp\{-\alpha w_0(1 + \underline{\tilde{x}}'\underline{\tilde{\mu}}) + \frac{\alpha^2 w_0^2}{2}\underline{x}'\Sigma\underline{x}\}$

$\square$

Under Assumption 2.1., the optimization problem (2.1) transforms into

$$\underset{\substack{\tilde{\underline{x}} \in \mathbb{R}^{n+1} \\ \tilde{\underline{x}}'\underline{e} = 1 \\ \tilde{\underline{x}}'\tilde{\underline{\mu}} = r}}{\tilde{\underline{x}}^* = \operatorname{argmax}} \; -\frac{1}{\alpha} \exp\Big\{ \underbrace{-\alpha w_0 (1 + \tilde{\underline{x}}'\underline{\mu})}_{\text{determined by side constraints}} + \frac{\alpha^2 w_0^2}{2} \underline{x}'\Sigma \underline{x} \Big\}$$

no tilda because of Σ

$$\Rightarrow \quad \tilde{\underline{x}}_r = \underset{\substack{\tilde{\underline{x}} \in \mathbb{R}^{n+1} \\ \tilde{\underline{x}}'\underline{e} = 1 \\ \tilde{\underline{x}}'\tilde{\underline{\mu}} = r}}{\operatorname{argmin}} \; \underline{x}'\Sigma \underline{x}$$

Markowitz problem (Mean-Variance optimization problem)

$$= \operatorname{argmin} \; \operatorname{Var}(\tilde{\underline{x}}'\tilde{\underline{R}})$$
$$\dots$$

Markowitz Problem

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$\tilde{\underline{R}} = (R_0, R_1, \dots, R_n)'$, $(n+1)$-dim vector of returns
with - R_0 risk-free return ($\mu_0 = \mathbb{E}(R_0) = R_0$)
- $\underline{R} = (R_1, \dots, R_n)$ the risky returns with μ and Σ

For given (required) expected return $r \in \mathbb{R}$, solve
$\tilde{\underline{x}}^* = \operatorname{argmin} \; x'\Sigma x$
under constraints: $\tilde{x} \in \mathbb{R}^{n+1}$, $\tilde{\underline{x}}'e = 1$, $\tilde{\underline{x}}'\tilde{\underline{\mu}} \geq r$

2.3. Optimization techniques

see book: Ingersoll

- $A \in \mathbb{R}^{n\times n}$ positive definite, if $\underline{x}'A\underline{x} > 0 \;\; \forall \underline{x} \neq \underline{0}$
- if A is positive definite, then A^{-1} exists and is also positive def.
- if $\underline{X}$ is a random vector with covariance matrix Σ,
  $\operatorname{Cov}(A\underline{X} + \underline{b}, C\underline{X} + \underline{d}) = A\Sigma C'$
- $f: \mathbb{R}^n \to \mathbb{R}$ twice differentiable $\underline{x} \mapsto f(\underline{x})$
  $\nabla f = \frac{\partial f}{\partial \underline{x}} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^n$, gradient
  $Hf = \frac{\partial^2 f}{\partial \underline{x}^2} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij} \in \mathbb{R}^{n\times n}$, Hessian

example: $A \in \mathbb{R}^{n \times n}$ $\quad \sum_i \sum_j a_{ij} x_i x_j$

Quadratic form $f(\underline{x}) = \underline{x}' A \underline{x}$

$$\left.\begin{aligned} \frac{\partial f}{\partial \underline{x}} &= (A + A')\,\underline{x} &&= 2A\underline{x} \\ H_f &= A + A' &&= 2A \end{aligned}\right\} \text{ if } A = A'$$

1) Unconstraint local maximum of $f: \mathbb{R}^n \to \mathbb{R}$

Solve $\nabla f = 0$, $\underline{z}'(H_f)\underline{z} < 0 \quad \forall \underline{z} \neq 0$

2) Local maximum with equality constraint (Method of Lagrange) $\max\limits_{g_i(\underline{x}) = a_i} f(\underline{x})$

$$\mathcal{L}(\underline{x}, \lambda) = f(\underline{x}) - \sum \lambda_i (g_i(\underline{x}) - a_i)$$

Solve $\begin{cases} \frac{\partial \mathcal{L}}{\partial \underline{x}} = \nabla f - \lambda \nabla g(\underline{x}) \overset{!}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} = -g_i(\underline{x}) + a_i \overset{!}{=} 0 \end{cases}$ + second order constraint.

3) Local maximum with an inequality constraint (Method of Kuhn-Tucker) $\max\limits_{g(x) \geq a} f(\underline{x})$

$$\mathcal{L}(\underline{x}, \lambda, b) = f(x) - \lambda\,(g(\underline{x}) - b)$$

Solve
$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = \nabla f - \lambda \nabla g(\underline{x}) \overset{!}{=} 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -g(\underline{x}) + b \overset{!}{=} 0$$
$$\frac{\partial \mathcal{L}}{\partial b} = \lambda < 0$$
$$(b - a)\lambda = 0, \quad b \geq a \qquad \text{+ second order constraint}$$

2.4. Mean-Variance Problem, without riskless asset.

Model Assumptions 2.3

Assume we have n risky assets with returns $\underline{R} = (R_1, \dots, R_n)'$ satisfying

A1) $\underline{\mu} = \mathbb{E}(\underline{R})$, $\exists i, j \in \{1, \dots, n\}: \mu_i \neq \mu_j$

A2) $\Sigma = Var(\underline{R})$ positive definite

A_1 implies $\underline{\mu} \neq \lambda \underline{e} \;\; \forall \lambda \in \mathbb{R}$ $\quad$ every position is achievable $\Rightarrow A^{-1}$ exists.

Terminology: $\underline{x} \in \mathbb{R}^n$ with $\underline{x}'\underline{e} = 1$ will be called investment strategy.

We define:

1) portfolio return: $R_p(\underline{x}) = \underline{x}'\underline{R}$
2) expected ptf return: $\mathbb{E}(R_p(\underline{x})) = \underline{x}'\underline{\mu}$
3) ptf variance: $Var(R_p(\underline{x})) = \underline{x}'\Sigma\underline{x}$

Definition 2.4. An investment strategy $\underline{z} \in \mathbb{R}^n$ is <u>efficient</u> if there is no other investment strategy $\underline{x} \in \mathbb{R}^n$ with

$$\mathbb{E}(R_p(\underline{z})) \leq \mathbb{E}(R_p(\underline{x}))$$
$$Var(R_p(\underline{z})) > Var(R_p(\underline{x}))$$

Assume r_p is given, then

$$x^+_{r_p} = \underset{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x}'\underline{e} = 1 \\ \underline{x}'\underline{\mu} \geq r_p}}{\arg\max} -\frac{1}{2}\underline{x}'\Sigma\underline{x} \qquad (2.2)$$

Kuhn-Tucker: $\mathcal{L}(\underline{x}, \lambda_1, \lambda_2, r) = -\frac{1}{2}\underline{x}'\Sigma\underline{x} - \lambda_1(\underline{x}'\underline{e} - 1) - \lambda_2(\underline{x}'\underline{\mu} - r)$

in order to solve (2.2), we consider

1) $\frac{\partial \mathcal{L}}{\partial \underline{x}} = -\Sigma\underline{x} - \lambda_1\underline{e} - \lambda_2\underline{\mu} \overset{!}{=} 0$

2) $\frac{\partial \mathcal{L}}{\partial \lambda_1} = -(\underline{x}'\underline{e} - 1) \overset{!}{=} 0$

3) $\frac{\partial \mathcal{L}}{\partial \lambda_2} = -(\underline{x}'\underline{\mu} - r) \overset{!}{=} 0$ + 2nd order constraint.
↳ ok in this case.

4) $\frac{\partial \mathcal{L}}{\partial r} = \lambda_2 \leq 0$

5) $(r - r_p)\lambda_2 = 0$, $r \geq r_p$

(A) in the first step, we only consider 1)-3) for a given r.

1') $\underline{x} = -\Sigma^{-1}(\lambda_1\underline{e} + \lambda_2\underline{\mu}) = -\Sigma^{-1}(\underline{e}, \underline{\mu})\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix}$

$\overset{2,3)}{\Longrightarrow} (\underline{e}, \underline{\mu})'\underline{x} = \begin{pmatrix}1 \\ r\end{pmatrix}$

$\Rightarrow \begin{pmatrix}1 \\ r\end{pmatrix} = (\underline{e}, \underline{\mu})'(-\Sigma^{-1})(\underline{e}, \underline{\mu})\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = A\begin{pmatrix}-\lambda_1 \\ -\lambda_2\end{pmatrix}$

with $A = \begin{pmatrix}\underline{e}'\Sigma^{-1}\underline{e}, & \underline{\mu}'\Sigma^{-1}\underline{e} \\ \underline{e}'\Sigma^{-1}\underline{\mu}, & \underline{\mu}'\Sigma^{-1}\underline{\mu}\end{pmatrix} = \begin{pmatrix}a & b \\ b & c\end{pmatrix}$

Claim: A is positive definite

and then $\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ r \end{pmatrix}$

$\Rightarrow$ $\underline{\underline{x}_r = \Sigma^{-1}(\underline{e}, \underline{\mu}) A^{-1}\begin{pmatrix} 1 \\ r \end{pmatrix}}$ optimal portfolio

proof of claim: $\underline{z} = (z_1, z_2)' \neq \underline{0}$: $\underline{z}' A \underline{z} > 0$?

$\underline{y} = (\underline{e}, \underline{\mu})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underline{e} z_1 + \underline{\mu} z_2 \neq \underline{0} \quad \forall z \neq 0 \text{ (A1)}$

$\underline{z}' A \underline{z} = \underline{y}' \Sigma^{-1} \underline{y} > 0$ because Σ^{-1} is positive definite.

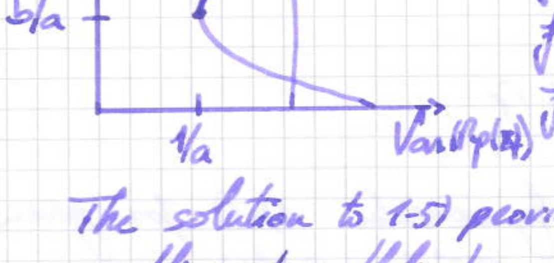
here $\mathbb{E}(R_p(\underline{x}_r)) = r$, with

$$\begin{aligned}\underline{Var}(R_p(\underline{x}_r)) &= \underline{x}_r' \Sigma \underline{x}_r = \\ &= (1,r) A^{-1} (\underline{e}, \underline{\mu})' \Sigma^{-1} \Sigma \Sigma^{-1} (\underline{e}, \underline{\mu}) A^{-1} (1,r)' \\ &= \underline{(1,r) A^{-1} (1,r)'}\end{aligned}$$

$\Rightarrow$ the optimal portfolio satisfying 1)-3) has the properties

$$\left|\begin{aligned} &\underline{x}_r = \Sigma^{-1}(\underline{e}, \underline{\mu}) A^{-1}\begin{pmatrix} 1 \\ r \end{pmatrix} \\ &A^{-1} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \\ &\mathbb{E}(R_p(\underline{x}_r)) = r \\ &Var(R_p(\underline{x}_r)) = \frac{c-2br+ar^2}{ac-b^2} \checkmark \end{aligned}\right.$$

Note: From Σ positive definite $\Rightarrow ac-b^2>0$
(Cauchy-Schwarz inequality)



$f(x) = c - 2bx + ax^2$

$f'(x) = -2b + 2ax = 0 \Rightarrow x = b/a$

$f''(x) = 2a = 2\underline{e}'\Sigma^{-1}\underline{e} > 0$

$f(b/a) = 1/a \cdot \underbrace{(ca-b^2)}_{\text{then cancels out}}$

The solution to 1-5) provides that for $r \le b/a$, the efficient portfolio has return b/a

Define the <u>global minimum variance portfolio</u> by

$$r_{gmv} = b/a$$
$$\sigma^2_{gmv} = Var(R_p(\underline{x}_{gmv})) = 1/a$$
$$\underline{x}_{gmv} = \frac{1}{a}\Sigma^{-1}\underline{e} = \frac{1}{\underline{e}'\Sigma^{-1}\underline{e}}\Sigma^{-1}\underline{e} \quad \checkmark$$

Theorem: under Model Assumption 2.3 and for given $r_p \in \mathbb{R}$, the efficient portfolio $\underline{x}^+_{r_p} \in \mathbb{R}^n$ is given by
$$\underline{x}^+_{r_p} = \begin{cases} \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\binom{1}{r_p}, & \text{if } r_p \geq r_{gmv} \\ \underline{x}_{gmv}, & \text{otherwise} \end{cases}$$
$$= \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\binom{1}{\max\{r_p, r_{gmv}\}} \quad \Box$$

Definition 2.6: the optimal investment strategy $\underline{x}_r$ to 1)-3) for a given $r \in \mathbb{R}$ is called <u>minimum variance portfolio for r.</u>

Corollary 2.7:
- $\underline{x}_r = \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\binom{1}{r}$
- For $r \geq r_{gmv}$, $\underline{x}_r = x_r^+$.

<u>No riskless asset</u>

$$\underline{x}^+_{r_p} = \underset{\substack{x \in \mathbb{R}^n \\ \underline{x}'\underline{e} = 1 \\ \underline{x}'\underline{\mu} = r_p}}{\operatorname{argmin}} \ \underline{x}'\Sigma\underline{x}$$

efficient investment strategies:
$$\underline{x}^+_{r_p} = \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\binom{1}{\max\{r_p, r_{gmv}\}}$$
$$r_{gmv} = b/a, \quad \sigma^2_{gmv} = 1/a$$
$$\underline{x}_{gmv} = \frac{1}{\underline{e}'\Sigma^{-1}\underline{e}}\Sigma^{-1}\underline{e}, \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \underline{e}'\Sigma^{-1}\underline{e} & \underline{\mu}'\Sigma^{-1}\underline{e} \\ \underline{\mu}'\Sigma^{-1}\underline{e} & \underline{\mu}'\Sigma^{-1}\underline{\mu} \end{pmatrix}$$

minimum variance portfolios: $\underline{x}_r = \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\binom{1}{r}$, $r \in \mathbb{R}$.

Corollary 2.8

a) every minimum variance portfolio is a linear combination of the two portfolios

$$\underline{x}_{gmv} \quad \text{and} \quad \underline{x}^{(0)} = \frac{1}{\underline{e}'\Sigma^{-1}\underline{\mu}} \Sigma^{-1}\underline{\mu}$$

b) every linear combination of $\underline{x}_{gmv}$ and $\underline{x}^{(0)}$ is a minimum variance portfolio

proof: from the proof of thm 2.5, we know

lagrange $\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ r \end{pmatrix} \Rightarrow \underline{x}_r = \Sigma^{-1}(\underline{e},\underline{\mu})\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}$

$$\underline{x}_r = -\lambda_1 \Sigma^{-1}\underline{e} - \lambda_2 \Sigma^{-1}\underline{\mu}$$

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{1}{ac-b^2}\begin{pmatrix} c-br \\ -b+ar \end{pmatrix}$$

i) for $r = b/a \Rightarrow \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 1/a \\ 0 \end{pmatrix} \Rightarrow r_{gmv}$

ii) $r = c/b \overset{def}{=} r^{(0)} \Rightarrow \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/b \end{pmatrix}$

$$\Rightarrow \underline{x}^{(0)} := \underline{x}_{r^{(0)}} = \frac{1}{\underline{\mu}'\Sigma^{-1}\underline{e}} \Sigma^{-1}\underline{\mu}$$

$\Rightarrow \underline{x}_{gmv}$ and $\underline{x}^{(0)}$ are minimum variance portfolios

Note: since $\underline{\mu} \neq \lambda\underline{e} \quad \forall \lambda \in \mathbb{R}$ (assumption A1)

$$\underline{x}_{gmv} \neq \underline{x}^{(0)}, \quad r_{gmv} \neq r^{(0)}$$

a) Choose $r \in \mathbb{R} \Rightarrow \exists \alpha : r = \alpha r^{(0)} + (1-\alpha) r_{gmv}$

$$\Rightarrow \underline{x}_r = \Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\begin{pmatrix} \alpha + (1-\alpha) \\ \alpha r^{(0)} + (1-\alpha) r_{gmv} \end{pmatrix}$$

$$= \alpha\Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\begin{pmatrix} 1 \\ r^{(0)} \end{pmatrix} + (1-\alpha)\Sigma^{-1}(\underline{e},\underline{\mu})A^{-1}\begin{pmatrix} 1 \\ r_{gmv} \end{pmatrix}$$

$\Rightarrow \underline{x}_r = \alpha\,\underline{x}^{(0)} + (1-\alpha)\,\underline{x}_{gmv} \quad \Rightarrow a$

b) like for a) but start at the end.

$\square$

Definition 2.9

Two investment strategies $\underline{x}$ and y are orthogonal if $Cov(R_p(\underline{x}), R_p(y)) = 0$.

Proposition 2.10

For every minimum variance portfolio $\underline{x}_r$ with $r \neq r_{gmv}$ there is a unique orthogonal minimum portfolio $\underline{x}_r{}^{\perp}$

Furthermore, $r^{\perp} = \dfrac{c - br}{b - ar}$

proof: Note $r \neq r_{gmv} = b/a \Rightarrow r^{\perp}$ well defined.

$$
\begin{aligned}
0 &\overset{?}{=} Cov(R_p(\underline{x}_r{}^{\perp}), R_p(\underline{x}_r)) \\
&= (1, r^{\perp}) A^{-1} \underbrace{(\underline{e}, \underline{\mu}) \Sigma^{-1} \Sigma \Sigma^{-1} (\underline{e}, \underline{\mu})}_{A} A^{-1} \binom{1}{r} \\
&= (1, r^{\perp}) \; A^{-1} \binom{1}{r} \\
&= \frac{1}{ac - b^2} (1, r^{\perp}) \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \binom{1}{r} \\
&= \frac{1}{ac - b^2} (1, r^{\perp}) \begin{pmatrix} c - br \\ -b + ar \end{pmatrix} \\
&= \frac{1}{ac - b^2} (c - br - r^{\perp} b + r^{\perp} ar) \overset{!}{=} 0.
\end{aligned}
$$

$$\Rightarrow r^{\perp} = \frac{c - br}{b - ar}$$

□

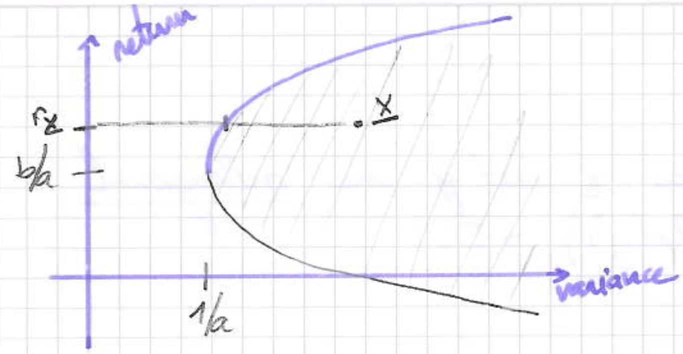
Theorem 2.11. CAPM (Capital asset Pricing Model) no riskless asset.

Assume that $\underline{x}_r$ is a minimum variance portfolio with $r \neq r_{gmv}$ and $\underline{x}_{r^{\perp}}$ is the corresponding (unique) orthogonal minimum variance portfolio.

Choose an arbitrary investment strategy $\underline{x} \in \mathbb{R}^n$ which provides return $r_{\underline{x}} = \underline{x}' \underline{\mu}$

$$\therefore \quad r_{\underline{x}} - r^{\perp} = \beta_{\underline{x}, r} (r - r^{\perp})$$

with $\beta_{\underline{x}, r} = \dfrac{Cov(\underline{x}' \underline{R}, \underline{x}_r' \underline{R})}{Var(\underline{x}_r' \underline{R})}$



Remark: if $Cov(\underline{x}'\underline{R}, \underline{x}_r'\underline{R})$ is small
then $\beta_{\underline{x},r}$ is small
=> expected return of $\underline{x}$ given by $r_{\underline{x}} - r^\perp$ is small
=> the price for $\underline{x}$ at time 0 is high
=> look for pf with low correlation -> diversification.

proof: $\underline{e}_j = (0, \dots, 0, 1, 0, \dots, 0)' \in \mathbb{R}^n$
notation $\underline{x}'R = R_p(\underline{x})$ $\quad R_j = Re_j$

$$Cov(R_j, R_p(\underline{x}_r)) = \underline{e}_j'\Sigma\,\underline{x}_r = \cancel{\underline{e}_j'\Sigma(-\lambda_1\Sigma^{-1}\underline{e} - \lambda_2\Sigma^{-1}\underline{\mu})}$$

$= e_j'\Sigma\Sigma^{-1}(e,\mu)A^{-1}\binom{1}{r}$

$= e_j'(e,\mu)A^{-1}\binom{1}{r}$

$= (1,\mu_j)A^{-1}\binom{1}{r}$

$$= \cancel{\underline{e}_j'(-\lambda_1\underline{e} - \lambda_2\underline{\mu})} = -\lambda_1 - \lambda_2\mu_j$$
$$= \cancel{(1,\mu_j)\binom{-\lambda_1}{-\lambda_2}} = (1,\mu_j)A^{-1}\binom{1}{r}$$

prop 2.10: $0 = Cov(R_p(\underline{x}_{r^\perp}), R_p(\underline{x}_r))$
$$= \underline{x}_{r^\perp}\Sigma\,\underline{x}_r = (1, r^\perp)A^{-1}\binom{1}{r}$$

$$Cov(R_j, R_p(\underline{x}_r)) = Cov(R_j, R_p(\underline{x}_r)) - Cov(R_p(\underline{x}_{r^\perp}), R_p(\underline{x}_r))$$

1) calc $Cov(R_j, x_r'R)$ (linearity)
2) $Cov(\underline{x}'R, x_r'R)$
3) $Cov(x_r'R, x_r'R)$ as special case.
done □

$$= (0, \mu_j - r^\perp)A^{-1}\binom{1}{r}$$
$$= (0, \mu_j - r^\perp)\boxed{\frac{1}{ac-b^2}\binom{c-br}{-b+ar}} = A^{-1}\binom{1}{r}$$
$$= \frac{(\mu_j - r^\perp)(-b+ra)}{ac-b^2}$$

Covariance of one single return?

By linearity $Cov(\underline{x}'\underline{R}, R_p(\underline{x}_r)) = \dfrac{(\underline{x}'\underline{\mu} - r^\perp)(-b+ra)}{ac-b^2}$
$$= \frac{(r_{\underline{x}} - r^\perp)(-b+ra)}{ac-b^2}$$

$$Var(R_p(\underline{x}_r)) = Cov(R_p(\underline{x}_r), R_p(\underline{x}_r))$$
$$= \frac{(r-r^\perp)(-b-ra)}{ac-b^2}$$

$$\Rightarrow \frac{Cov(\underline{x}'\underline{R}, R_p(\underline{x}_r))}{(r_x - r^\perp)} = \frac{Var(R_p(\underline{x}_r))}{r-r^\perp}$$

$$\Rightarrow (r_x - r^\perp) = \beta_{\underline{x},r}(r - r^\perp)$$

□

2.5. Mean-Variance Optimization with a risk free asset.

Assumptions 2.12.

Assume we have $n+1$ assets:

- $R_0 = \mu_0$ risk-free asset
- n risky assets $\underline{R} = (R_1, \dots, R_n)$ with
  - A1) $\exists j \in \{1, \dots, n\} : \mu_j = \mathbb{E}(R_j) \neq \mu_0$
  - A2) $\underline{R}$ has a positive definite covariance matrix Σ.

$\underline{\tilde{x}} \in \mathbb{R}^{n+1}$ investment strategy, $\underline{\tilde{x}}'\underline{\tilde{e}} = 1$

$$R_p(\underline{\tilde{x}}) = \sum_{i=0}^{n} x_i R_i = \sum_{i=0}^{n} x_i (R_i - \mu_0) + \sum x_i \mu_0$$
$$= \sum_{i=1}^{n} x_i R_i^e + \mu_0$$

where $R_i^e = R_i - \mu_0$ excess return of asset i.

Now: it will be sufficient to study excess returns $\underline{R}^e$.

optimal
$$\underline{\tilde{x}}^+_{r_p} = \underset{\substack{\underline{\tilde{x}} \in \mathbb{R}^{n+1} \\ \underline{\tilde{x}}'\underline{\tilde{e}} = 1 \\ \underline{\tilde{x}}'\underline{\tilde{\mu}} \geq r_p}}{\operatorname{argmin}} \underline{x}'\Sigma\underline{x} \iff \boxed{\underline{\tilde{x}}_{r_p} = \underset{\substack{\underline{\tilde{x}} \in \mathbb{R}^{n+1} \\ \underline{\tilde{x}}'\underline{\tilde{e}} = 1 \\ \underline{x} \cdot \underline{\mu}^e \geq r_p^e}}{\operatorname{argmin}} \underline{x}'\Sigma\underline{x}}$$

$$\iff \underline{x}'\underline{\mu}^e + \mu_0 \geq r_p$$
$$\iff \underline{x}\,\underline{\mu}^e \geq \underline{r_p - \mu_0 = r_p^e}$$

$$\underline{\tilde{x}}'\underline{\tilde{\mu}} = \mathbb{E}(R_p(\underline{\tilde{x}})) = \sum_{i=1}^{n} x_i \underbrace{\mathbb{E}(R_i^e)}_{\mu_i^e} + \mu_0$$
$$= \underline{x}'\underline{\mu}^e + \mu_0$$

$$\Rightarrow \quad \underline{x}_{r_p}^+ = \underset{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x}'\underline{\mu}^e \geq r_p^e}}{\operatorname{argmin}} \underline{x}'\Sigma\underline{x}$$

no 1st coordinate

$\underline{\mu} = \mu_0 \underline{e} + \underline{\mu}^e$

$x_0 = 1 - \underline{x}_r'e$

in order to have normalization

$x_0^+ = 1 - \underline{x}_{rp}^{+\,\prime} \underline{e}$

$\Rightarrow \underline{\tilde{x}}_{rp} = (x_0^+ , \underline{x}_{rp}^+)$

to solve this optimization problem

$$\underline{x}_r = \arg \max_{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x}'\underline{\mu}^e = r^e}} -\frac{1}{2} \underline{x}' \Sigma \underline{x}$$

Lagrange $\mathcal{L}(\underline{x}, \lambda) = -\frac{1}{2} \underline{x}' \Sigma \underline{x} - \lambda(\underline{x}'\underline{\mu}^e - r^e)$

$$\left.\begin{aligned} &a) \ \frac{\partial \mathcal{L}}{\partial x} = -\Sigma \underline{x} - \lambda \mu^e \overset{!}{=} 0 \\ &b) \ \frac{\partial \mathcal{L}}{\partial \lambda} = -(x'\mu^e - r^e) \overset{!}{=} 0 \end{aligned}\right\} \text{ to solve}$$

$\Rightarrow \underline{x} = -\lambda \Sigma^{-1} \underline{\mu}^e \quad (\exists j : \mu_j \neq \mu_0 \Rightarrow \mu^e \neq 0)$

$$r^e = (\underline{\mu}^e)'(-\lambda \Sigma^{-1} \underline{\mu}^e) = -\lambda \underbrace{\left((\mu^e)' \Sigma^{-1} \underline{\mu}^e\right)}_{>0}$$

$$\Rightarrow -\lambda = \frac{r^e}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e}$$

$$\Rightarrow \underline{x}_r = \frac{r^e}{(\mu^e)' \Sigma^{-1} \underline{\mu}^e} \Sigma^{-1} \cdot \underline{\mu}^e$$

$x_0 = 1 - \underline{x}_r' e.$

Proposition 2.13

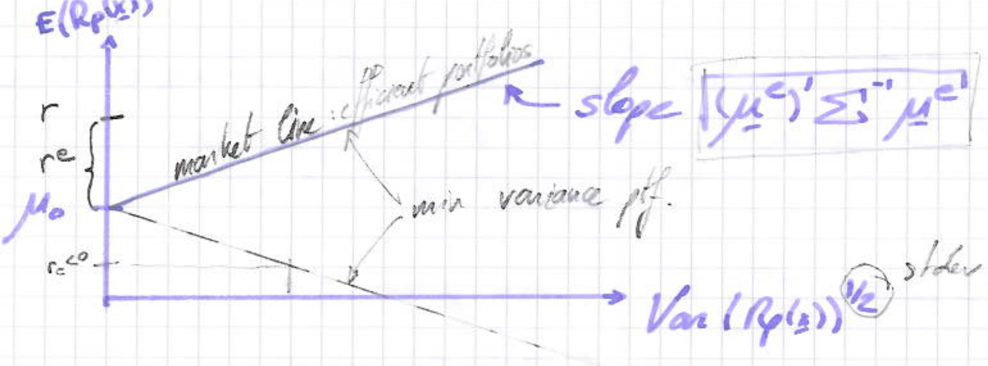
The minimum variance portfolio $r^e \in \mathbb{R}$ is given $\underline{x}_{r^e} = \frac{r^e}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e} \Sigma^{-1} \underline{\mu}^e$

recall A1) $\underline{\mu} \neq \lambda \underline{e} \ \ \forall \lambda \in \mathbb{R} \Rightarrow \underline{\mu}^e \neq \underline{0}$

A2) Σ positive definite.

\+ riskless asset.

$$\begin{aligned} Var(R_p(\underline{x}_r)) &= \underline{x}_r' \Sigma \underline{x}_r \\ &= \left(\frac{r^e}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e} \right)^2 \underline{\mu}^{e\prime} \Sigma^{-1} \Sigma \Sigma^{-1} \underline{\mu}^e \\ &= \frac{(r^e)^2}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e} \end{aligned}$$



Theorem 2.14.

Under Model assumptions 2.12., the efficient portfolios are given by

$$\underline{x}_r^+ = \frac{\max\{r^e, 0\}}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e} \Sigma^{-1} \underline{\mu}^e$$

$$x_0^+ = 1 - (x_r^+)' \underline{e}$$

proof: either we solve the corresponding Kuhn-Tucker problem, or we use "a graphical proof" □

Definition 2.15.

The minimum variance $\underline{x}_{tan}$ is called tangential portfolio, if we only invest into the risky assets, $\underline{x}_{tan}' \underline{e} = 1$. Denote the corresponding return by r_{tan}.

Proposition 2.16.

Assume $r_{gmv} \neq \mu_0$. Then there exists a unique tangential portfolio $\tilde{\underline{x}}_{tan}$ with

$$r^e_{tan} = \frac{(\mu^e)'\,\Sigma^{-1}\,\mu^e}{(\mu^e)'\,\Sigma^{-1}\,\underline{e}}$$

note: $r^e_{tan} \neq 0$ otherwise $x_{tan} = 0$ and $x_{tan}'\underline{e} \neq 1$

$$\underline{x}_{tan} = \frac{r^e_{tan}}{(\underline{\mu}^e)'\,\Sigma^{-1}\underline{\mu}^e}\,\Sigma^{-1}\underline{\mu}^e$$

proof: if the tangential pf exists, then it needs to be of the form

$$\underline{x}_r = \frac{r^e}{(\underline{\mu}^e)'\Sigma^{-1}\underline{\mu}^e}\cdot\Sigma^{-1}\underline{\mu}^e$$

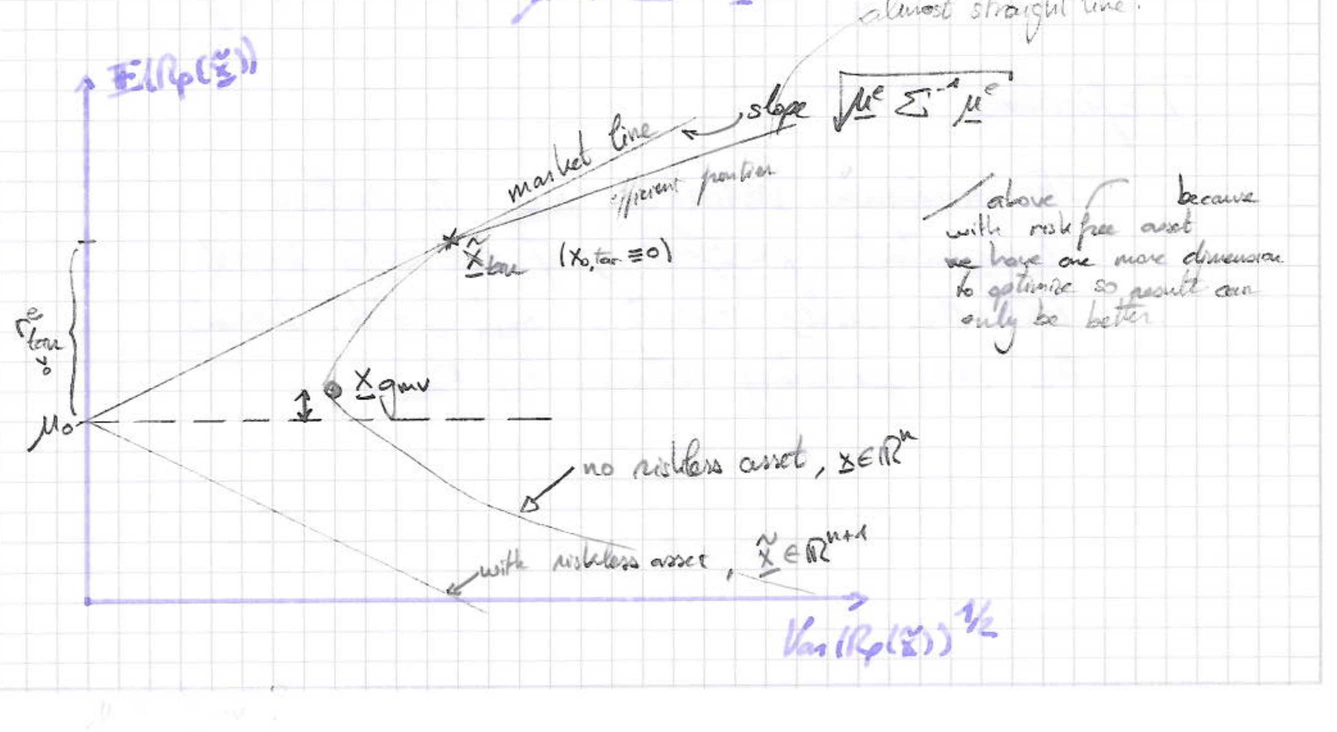
for some $r^e \in \mathbb{R}$. see prop. 2.13.

$$\underline{e}'\underline{x}_r = \frac{r^e}{(\underline{\mu}^e)'\Sigma^{-1}\underline{\mu}^e}\,\underline{e}'\Sigma^{-1}\underline{\mu}^e \overset{!}{=} 1$$

first check $\neq 0$ $\Rightarrow r^e\,(\underline{\mu}^e)'\Sigma^{-1}\underline{e} \overset{!}{=} (\underline{\mu}^e)'\Sigma^{-1}\underline{\mu}^e > 0.$

$$\begin{aligned}(\underline{\mu}^e)'\Sigma^{-1}\underline{e} &= (\underline{\mu}^e + \mu_0\underline{e} - \mu_0\underline{e})'\Sigma^{-1}\underline{e} \\ &= \underline{\mu}'\Sigma^{-1}\underline{e} - \mu_0\,\underline{e}\,\Sigma^{-1}\underline{e} \\ &= \underline{e}'\Sigma^{-1}\underline{e}\left(\underbrace{\frac{\underline{\mu}'\Sigma^{-1}\underline{e}}{\underline{e}\,\Sigma^{-1}\underline{e}}}_{=r_{gmv}} - \mu_0\right) \\ &= \underline{e}'\Sigma^{-1}\underline{e}\,(r_{gmv} - \mu_0) \neq 0.\end{aligned}$$

$$\Rightarrow \quad r^e_{tan} = \frac{(\underline{\mu}^e)'\,\Sigma^{-1}\underline{\mu}^e}{(\underline{\mu}^e)'\,\Sigma^{-1}\underline{e}} \qquad \square$$



Recall: we have parameters $\tilde{\underline{\mu}} \in \mathbb{R}^{n+1}$ and $\tilde{\Sigma} \in \mathbb{R}^{n \times n}$

If parameters are estimated / chosen

A) $\mu_0 < r_{gmv}$ (economically meaningful)

$$r_{gmv} = \frac{\tilde{\underline{e}}'\tilde{\Sigma}^{-1}\tilde{\underline{\mu}}}{\tilde{\underline{e}}'\tilde{\Sigma}^{-1}\tilde{\underline{e}}} = \frac{\tilde{\underline{e}}'\tilde{\Sigma}^{-1}(\tilde{\underline{\mu}}^e + \mu_0\tilde{\underline{e}})}{\tilde{\underline{e}}'\tilde{\Sigma}^{-1}\tilde{\underline{e}}}$$

$$= \frac{\underline{e}'\Sigma^{-1}\underline{\mu}^e}{\underline{e}'\Sigma^{-1}\underline{e}} + \mu_0$$

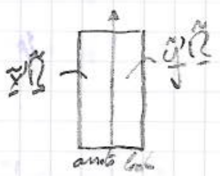
$$= \frac{(\underline{\mu}^e)'\Sigma^{-1}\underline{\mu}^e}{\underline{e}'\Sigma^{-1}\underline{e}} \cdot \frac{1}{r^e_{tan}} + \mu_0 > \mu_0$$

so $r_{gmv} > \mu_0 \iff r^e_{tan} > 0.$

2.6. Assets and Liability Management (ALM) example

Assume that we need to replicate a liability portfolio $\tilde{y} \in \mathbb{R}^{n+1}$ with $\tilde{y}'\tilde{\underline{e}} = 1$. and we would like to have an extra return $r_+ \in \mathbb{R}$

$$\tilde{\underline{x}}'\tilde{\underline{R}} - \tilde{y}'\tilde{\underline{R}}$$



$$\underline{x}^* = \underset{\substack{\tilde{\underline{x}} \in \mathbb{R}^{n+1} \\ \tilde{\underline{x}}'\tilde{\underline{e}} = 1 \\ \tilde{\underline{x}}'\tilde{\underline{\mu}} \geq \tilde{y}'\tilde{\underline{\mu}} + r_+}}{\operatorname{argmin}} \frac{1}{2} \operatorname{Var}(R_p(\tilde{\underline{x}}) - R_p(\tilde{y}))$$

$$\operatorname{Var}(R_p(\tilde{\underline{x}}) - R_p(\tilde{y})) = (\underline{x} - y)'\Sigma(\underline{x} - y)$$
$$= \underline{x}'\Sigma\underline{x} - 2y'\Sigma\underline{x} + \underbrace{y'\Sigma y}_{\text{given}}$$

$$\tilde{\underline{x}}'\tilde{\underline{\mu}} = \tilde{\underline{x}}\tilde{\underline{\mu}}^e + \tilde{\underline{x}}'(\mu_0\tilde{\underline{e}})$$
$$= \underline{x}'\underline{\mu}^e + \mu_0 \qquad (\mu_0^e = 0)$$
$$\tilde{y}'\tilde{\underline{\mu}} = y'\underline{\mu}^e + \mu_0$$

$$\Rightarrow \quad \underline{x}^* = \underset{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x}'\underline{\mu}^e \geq y'\underline{\mu}^e + r_+}}{\operatorname{argmin}} \frac{1}{2}(\underline{x}'\Sigma\underline{x} - 2y'\Sigma\underline{x})$$

$$x_0^* = 1 - \underline{x}^{*\prime}\underline{e}$$

$$\mathcal{L}(\underline{x},\lambda,r) = -\frac{1}{2}(\underline{x}'\Sigma_1\underline{x} - 2\underline{y}'\Sigma_1\underline{x}) - \lambda(\underline{x}'\underline{\mu}^e - (\underline{y}'\underline{\mu}^e + r))$$

Solve

a) $\frac{\partial\mathcal{L}}{\partial\underline{x}} = -\Sigma_1\underline{x} + \Sigma_1\underline{y} - \lambda\underline{\mu}^e \stackrel{!}{=} 0$

b) $\frac{\partial\mathcal{L}}{\partial\lambda} = -(\underline{x}'\underline{\mu}^e - (\underline{y}\underline{\mu}^e + r)) \stackrel{!}{=} 0$

c) $\frac{\partial\mathcal{L}}{\partial r} = \lambda \leq 0$ (Kuhn-Tucker)

d) $(r - r_+)\lambda = 0$, $r \geq r_+$

$\stackrel{a)}{\Rightarrow}$ $\underline{x} = \underline{y} - \lambda\Sigma_1^{-1}\underline{\mu}^e$

$\stackrel{b)}{\Rightarrow}$ $(\underline{y} - \lambda\Sigma_1^{-1}\underline{\mu}^e)'\underline{\mu}^e = \underline{y}'\underline{\mu}^e + r$

$\underline{y}'\underline{\mu}^e - \lambda(\underline{\mu}^e)'\Sigma_1^{-1}\underline{\mu}^e = \underline{y}'\underline{\mu}^e + r$

$\Rightarrow -\lambda = r/(\underline{\mu}^e)'\Sigma_1^{-1}\underline{\mu}^e$ r and λ have different signs

$\Rightarrow \underline{x} = \underline{y} + \frac{r}{(\underline{\mu}^e)'\Sigma_1^{-1}\underline{\mu}^e}\Sigma_1^{-1}\underline{\mu}^e$

$r_+ \in \mathbb{R}$

- $r_+ > 0 \Rightarrow r \geq r_+ > 0 \Rightarrow \lambda < 0 \Rightarrow r = r_+$
- $r_+ = 0$: $r > 0 \Rightarrow \lambda = 0 \Rightarrow r = 0$ ↯
  $\Rightarrow r = r_+ = 0 \Rightarrow \lambda = 0$
- $r_+ < 0$: $0 > r \geq r_+ \Rightarrow \lambda > 0$ ↯
  $\Rightarrow r \geq 0 \Rightarrow \lambda = 0 \Rightarrow r = 0.$

Theorem 2.18 (ALM problem)

Under 2.12, the solution to (ALM) is given by

$$\underline{x}^+ = \underline{y} + \frac{\max\{0, r_+\}}{(\underline{\mu}^e)'\Sigma_1^{-1}\underline{\mu}^e}\Sigma_1^{-1}\underline{\mu}^e$$

$$x_0^+ = 1 - (\underline{x}^+)'\underline{e}$$

$Var(R_P(\underline{x}^+)) = (\underline{x}^+)'\Sigma_1\underline{x}^+$

$Var(R_P(\underline{x}^+) - R_P(\underline{x})) = (\underline{x}^+ - \underline{y})'\Sigma_1(\underline{x}^+ - \underline{y})$

$= \max\{0, r_+\}^2/(\underline{\mu}^e)'\Sigma_1^{-1}\underline{\mu}^e$ ✓

Chapter III: Capital Asset Pricing Model (CAPM)

Recall prop. 2.16.

Lemma 3.1. Under Model Assumptions 2.12 and $r_{gmv} \neq \mu_0$, every minimum variance ptf $\tilde{\underline{x}}_{r^e} \in \mathbb{R}^{n+1}$ is a linear combination of the tangential portfolio $\tilde{\underline{x}}_{tan} \in \mathbb{R}^{n+1}$ and riskless portfolio $\tilde{\underline{x}}_0 = (1, 0, \dots, 0)' \in \mathbb{R}^{n+1}$

no tilde

proof: $r^e \in \mathbb{R}$. $\underline{x}_{r^e} = \dfrac{r^e \cdot r^e_{tan} / r^e_{tan}}{(\underline{\mu}^e)' \Sigma^{-1} \underline{\mu}^e} \Sigma^{-1} \underline{\mu}^e = \dfrac{r^e}{r^e_{tan}} \underline{x}_{tan}$

$$x_0 = 1 - (\underline{x}^e_r)' \underline{e} = 1 - \frac{r^e}{r^e_{tan}} \underbrace{\underline{x}_{tan}' \underline{e}}_{1} = 1 - r^e / r^e_{tan}$$

$$\tilde{\underline{x}}'_{r^e} = \left(1 - \frac{r^e}{r^e_{tan}}\right) \tilde{\underline{x}}_0 + \frac{r^e}{r^e_{tan}} \tilde{\underline{x}}_{tan} \qquad (3.1)$$

$\square$

Model Assumptions 3.2 (Financial Market)

- Supply: Assume we have $n+1$ financial assets satisfying MA. 2.12. with $r_{gmv} \neq \mu_0$. The value of asset $j = 0, \dots, n$ at time 0 is given by Π_j. (Market)
- Demand: We have $N \geq 1$ financial agents all holding a minimum variance portfolio $\tilde{\underline{x}}^{(i)}$, $i = 1, \dots, N$ and having initial wealth w_i. at time 0.

Assumption 3.3. (Economic Principle)

We assume that a risk exchange economy with market clearing. ie. supply = demand.

Supply: Let us denote the market portfolio of risky assets by $\underline{x}^M \in \mathbb{R}^n$ with

$$x_j^M = M_j / M_\cdot \qquad M_\cdot = \sum_j M_j$$

$\Rightarrow$ the value vector of risky assets is at time 0 given by $(M_1, \dots, M_n)' = M_\cdot \left(M_1/M_\cdot, \dots, M_n/M_\cdot \right)' = M_\cdot \underline{x}^M \in \mathbb{R}^n$

Demand: Every financial agent holds a minimum variance portfolio $\underline{x}^{\mu(i)}$, which correspond to his expected excess return r_i^E, $i = 1, \dots, N$.

$\Rightarrow$ Portfolio of agent: $w_i \left(1 - \frac{r_i^E}{r_{tan}^E}, \frac{r_i^E}{r_{tan}^E} x_{tan} \right)$ see 3.1

$\Rightarrow$ total demand of risky assets.

$$\sum_{i=1}^N w_i \, r_i^E / r_{tan}^E \, \underline{x}_{tan} = \sum_i w_i r_i^E / r_{tan}^E \cdot \underline{x}_{tan}$$

Market clearing condition: $M_\cdot \underline{x}^M \overset{!}{=} \frac{\sum_i w_i r_i^E}{r_{tan}^E} \underline{x}_{tan}$ $\Big|$ c!

$$\Rightarrow \quad M_\cdot = \sum_i \frac{w_i r_i^E}{r_{tan}^E}, \quad \underline{x}^M = \underline{x}_{tan} \, !$$

Theorem 3.4. (CAPM pricing formula)

Assume Model Assumption 3.2. and Assumption 3.3. Then,

$$\mu_j = \mu_0 + \beta_j (r_M - \mu_0), \quad j = 1, \dots, n$$

with $r_M = \mathbb{E}\left(R_p(\underline{x}^M) \right)$

$$\beta_j = \mathrm{Cov}\left(R_j, R_p(\underline{x}^M) \right) / \mathrm{Var}\left(R_p(\underline{x}^M) \right)$$

asset j — Market pff.

proof: Note $\underline{x}^M = \underline{x}_{tan}$, $\underline{e}_j = (0, \dots, 0, 1, 0, \dots, 0)' \in \mathbb{R}^n$

plug in values and check right.

$$\mathrm{Cov}\left(R_j, R_p(\underline{x}^M) \right) = \mathrm{Var}\left(R_p(\underline{x}^M) \right) / r_{tan}^E \; \mu_j^E \; \checkmark$$

$$\Rightarrow \mu_j^E = \mathrm{Cov} / \mathrm{Var} \; r_{tan}^E.$$

$$r_{tan}^E = \mathbb{E}\left(R_p(\underline{x}_{tan}) \right) - \mu_0 = r_M - \mu_0$$

$$\Rightarrow \mu_j - \mu_0 = \beta_j (r_M - \mu_0) \qquad \square$$

Remarks:

1) The expected returns μ_j can be viewed as "equilibrium returns": assume expected return of asset $j > \mu_j$, then price for asset j is too low, demand will increase, price will increase, expected return decreases, until $=$.

2) In general, μ_j is rather difficult to be estimated from data. Calibrate μ_j with CAPM formula:
   - μ_0 government bond.
   - r_M in general easier than determining μ_j
   - β_j can be estimated, more stable than μ_j.

3) Weaknesses of CAPM
   a) market clearing $\Rightarrow$ closed economy.
   b) all financial agents are mean-variance optimizers.
   c) the formula only holds true if every agent uses the same Σ!

Chapter IV: Arbitrage Pricing Theory (APT)

CAPM: $\mu_j = \mu_0 + \beta_j (r_M - \mu_0)$

idiosyncratic risk

$$\Rightarrow \underbrace{R_j}_{\mathbb{E}R_j = \mu_j} = \mu_0 + \beta_j \big(\underbrace{R_p(\underline{x}^M)}_{\mathbb{E}R_p(\underline{x}^M) = r_M} - \mu_0 \big) + \underbrace{\varepsilon_j}_{\mathbb{E}\varepsilon_j = 0}$$

4.1. Exact APT (no idiosyncratic risk)

Model 4.1.
- μ_0 is riskfree return
- $R_1, \dots, R_n$ risk returns with $R_i = \mu_i + \sum_{k=1}^{K} b_{ik} F_k$ with $\mathbb{E}F_k = 0$, $Cov(\underline{F}) = \Phi$
- $n > K$ allow to identify the K factors

key here.

Matrix Notation: $\boxed{\underline{R} = \underline{\mu} + B\underline{F}}$ with $B = (b_{ik})_{\substack{i=1,\dots,n\\k=1,\dots,K}}$

Since $n > K$, there is a portfolio $\underline{0} \neq \underline{x} \in \mathbb{R}^n$ that is orthogonal to all the column vectors of B $\quad B\underline{x} = 0$

$x_0 = 1 - \underline{x}'\underline{e}$

$$\Rightarrow \mathbb{E}(R_p(\tilde{\underline{x}})) = \tilde{\underline{x}}'\tilde{\underline{\mu}} = \mu_0 + \underline{x}'\underline{\mu}^e$$

$$Var(R_p(\tilde{\underline{x}})) = \underline{x}' Cov(\underline{R})\, \underline{x} = \underline{x}' Cov(\underline{\mu} + B\underline{F})\, \underline{x}$$
$$= \underline{x}' \underline{B}\, Cov(\underline{F})\, B'\underline{x} = (\underbrace{B'x}_{=0})' \,\phi\, (\underbrace{B'x}_{=0}) = 0.$$

$\underline{x}$ is a risk-free portfolio.

Example: $K = 1$, $n = 2$. $\quad b_1 \neq 0 \neq b_2 \quad b_1 \neq b_2$

$R_i = \mu_i + b_i F \quad i = 1,2.$

Choose a ptf $\underline{x} = (x_1, x_2)' = (x, 1-x)' \in \mathbb{R}^2$ s.t.

$Var(R_p(\underline{x})) = 0$ for some $x \in \mathbb{R}$.

$$R_p(\underline{x}) = xR_1 + (1-x)R_2 = x(\mu_1 + b_1F) + (1-x)(\mu_2 + b_2F)$$
$$= x\mu_1 + (1-x)\mu_2 + (\underbrace{xb_1 + (1-x)b_2}_{=0})F$$
$$= x\mu_1 + (1-x)\mu_2 \overset{!}{=} \mu_0 \leftarrow \text{arbitrage argument}$$

if $x = -b_2/b_1 - b_2$

Definition 4.2 : $\tilde{\underline{x}} \in \mathbb{R}^{n+1}$ is called an <u>arbitrage portfolio</u> if:

i) $\underline{x}'\underline{e} = -x_0$ (net investment 0)

ii) $\mathbb{E}(R_p(\tilde{\underline{x}})) > 0$ (positive return)

iii) $Var(R_p(\tilde{\underline{x}})) = 0$ (no risk)

$\Rightarrow R_p(\tilde{\underline{x}}) > 0$ $\mathbb{P}$-a.s. $\Rightarrow$ without any downside risk the arb. ptf. generates value "out of nothing".

Theorem 4.~~1~~3

Under assumptions 4.1 and under "no arbitrage assumption" we obtain:

$\exists \underline{\lambda} = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K$ (risk premium)

such that $\mu_i = \mu_0 + \sum_{k=1}^{K} b_{i,k} \lambda_k \quad \forall i = 1, \dots, n$

explanation: $\mu_i + \sum_{k=1}^{K} b_{i,k} F_k = R_i$

proof: choose $\underline{0} \neq \underline{x} \in \mathbb{R}^n$ with $B'\underline{x} = 0$.

$x_0 = \cancel{1} - \underline{x}'\underline{e}$, $\mathbb{E}(R_p(\tilde{\underline{x}})) = \cancel{\mu_0} + (\underline{\mu}^e)'\underline{x}$, $Var(R_p(\tilde{\underline{x}})) = 0$.

claim: under no-arbitrage $(\underline{\mu}^e)'\underline{x} = 0$

Assume $(\underline{\mu}^e)'\underline{x} \neq 0$, w.l.o.g. $(\underline{\mu}^e)'\underline{x} > 0$.

$\Rightarrow r = \mathbb{E}(R_p(\tilde{\underline{x}})) = \mu_0 + (\underline{\mu}^e)'\underline{x} > \mu_0$

- Invest 1 unit into the ptf $\tilde{\underline{x}}$
- Invest -1 unit into the riskless ptf $\tilde{y} = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$

then the net investment is 0. (i)

$\tilde{\underline{z}} := \tilde{\underline{x}} - \tilde{y}$, $Var(R_p(\tilde{\underline{z}})) = Var(R_p(\tilde{\underline{x}} - \tilde{y}))$

$= Var(R_p(\tilde{\underline{x}})) = 0$ (iii)

$\mathbb{E}(R_p(\tilde{\underline{z}})) = \mathbb{E}(R_p(\tilde{\underline{x}})) - \mathbb{E}(R_p(\tilde{y}))$

$= r - \mu_0 > 0$ (ii)

$\Rightarrow$ arbitrage ⚡ $\Rightarrow (\underline{\mu}^e)'\underline{x} = 0$.

$B: n \times k$

So $\underline{x}'\underline{\mu}^e = \underline{x}'(\underline{\mu} - \mu_0\underline{e}) = 0, \forall \underline{x}$ with $B'\underline{x} = \underline{0}$ $\qquad \underset{n}{B'\underline{x}}$ but $\underset{k}{B\lambda}$.

$\Rightarrow \underline{\mu} - \mu_0\underline{e}$ is in the span of the column vectors of B.

$\Rightarrow \exists \underline{\lambda} \in \mathbb{R}^K$ st $B\lambda = \underline{\mu} - \mu_0\underline{e}$ ✓

$\Rightarrow \underline{\mu} = \mu_0\underline{e} + B\underline{\lambda}$ $\square$

$\underline{\mu} - \mu_0\underline{e}$ is a lin comb of column vectors of B with factors λ.

Remarks:

- $n > K$: perfect description of the risk factors $\underline{F}$ by risky returns $\underline{R}$.
- every risk factor F_k obtains its arbitrage-free mean λ_k
- $\underline{K=1}$: $R_i = \mu_i + b_i F \Rightarrow \mu_i = \mu_0 + b_i \lambda$, $\lambda \in \mathbb{R}$

APT-formula

$\underline{\text{CAPM}}$: $\mu_i = \mu_0 + \beta_i (r_M - \mu_0)$

$\Rightarrow$ basically CAPM and APT model (for $K=1$) provide the same pricing formula for risky assets. Therefore, they are usually identified in practice, BUT THIS SHOULD NOT BE DONE!
Because these two formulae are based on completely different economic principles:
CAPM: market clearing (closed markets)
APT: no-arbitrage

$\Rightarrow$ statistical Tests cannot distinguish these two ~~models~~ formula models.

4.2 APT with idiosycratic risk

Model assumptions 4.4

- μ_0 risk-free asset/return
- infinite sequence of risky assets with returns $R_1, R_2, \ldots$ satisfying: $\underline{R}^N = (R_1, \ldots, R_N)'$ has for all $N \geq 1$ the following form:

$$\underline{R}^N = \underline{\mu}^N + B^N \underline{F} + \underline{\varepsilon}^N$$

- $\underline{F} = (F_1, \ldots, F_K)' \in \mathbb{R}^K$ with $\mathbb{E} F_k = 0$, $\text{Cov}(\underline{F}) = \Phi$ pos. def.
- B^N are the N first rows of a $(\infty \times K)$ matrix B.
- $\underline{\varepsilon}^N = (\varepsilon_1, \ldots, \varepsilon_N)'$ are the first N components of an infinite sequence $\varepsilon_1, \varepsilon_2, \ldots$, with $\mathbb{E}\varepsilon_i = 0$ $\forall i = 1, \ldots$
$\text{Cov}(\underline{\varepsilon}^N) = \Omega^N$ pos. def. and all eigenvalues of Ω^N are uniformly bounded (in N) by $\bar{\lambda}$.
- $\text{Cov}(F_k, \varepsilon_i) = 0$ $\forall i, k$

Definition 4.5: An <u>asymptotic arbitrage opportunity</u> is a sequence of portfolios $\underline{w}^N \in \mathbb{R}^N$ with

limsup because I can then select a subsequence that converges to earnings.

i) $(\underline{w}^N)' \underline{e}^N = 0$ (net investment 0)

ii) $\limsup_{N \to \infty} \mathbb{E}((\underline{w}^N)' R^N) \geq \delta > 0$

iii) $\lim_{N \to \infty} \mathrm{Var}((\underline{w}^N)' R^N) = 0.$

<u>Theorem 4.6</u>

Under Model Assumption 4.4 and the exclusion of asymptotic arbitrage, there exists a sequence $\underline{\lambda}^N = (\lambda_0^N, \dots, \lambda_K^N)$ such that for $N > K$

$$\mu_i = \lambda_0^N + \sum_{k=1}^K b_{j,k} \lambda_k^N + \underbrace{v_i^N}_{\text{residuals, error terms}} \quad i = 1, \dots, N$$

and $$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N (v_i^N)^2 = 0. \qquad (4.1)$$

proof: Multivariate linear regression:

a) $K=1$. goal: $\mu_i = \lambda_0^N + b_{i,1} \lambda_1^N + v_i^N \quad \forall i = 1, \dots, N$ to minimize 4.1.

Optimal least squares (OLS)

b) $K \geq 1$. $\underline{\mu}^N = \lambda_0^N \underline{e} + B^N \underline{\lambda}^N + \underline{v}^N$

$\Rightarrow \underline{v}^N = \underline{\mu}^N - (\lambda_0^N \underline{e} + B^N \lambda^N)$

find $\underline{\lambda}^N$ s.t. $\|v^N\|_2^2 = (v^N)'(v^N) = \sum_{i=1}^N (v_i^N)^2$ is minimal

$\Rightarrow$ OLS solve:

by parti. deriv. $\begin{cases} 1) \sum_{i=1}^N v_i^N = 0 \Rightarrow (v^N)' e = 0 \\ 2) \sum_{i=1}^N v_i^N b_{ik} = 0 \quad \forall k = 1, \dots, K \Rightarrow \underbrace{(B^N)'}_{K \times N} \underbrace{v^N}_{N \times 1} = \underline{0} \end{cases} \quad (4.2)$

So choose the solution $\underline{\lambda}^N$ to (4.2) and define the portfolio

$$\underline{w}^N = \frac{1}{\sqrt{N}} \underbrace{\frac{1}{\|v^N\|_2}}_{\text{normalization}} \underline{v}^N$$

$\longrightarrow \underline{0}$ as $N \to \infty$, but is $> \underline{0}$ as long as $N < \infty$, and we use it against Arbitrage.

This is a "ptf that is a candidate to arbitrage.

$\hookrightarrow$ proof by $(\neg B \Rightarrow \neg A)$

$\cdot$ (i) $(\underline{w}^N)'\underline{e}^N = \frac{1}{\sqrt{N}} \frac{1}{\|\underline{v}\|_2} \underbrace{(\underline{v}^N)'\underline{e}^N}_{=\sum v_i^N = 0\ (4.2)(1)} \underline{\underline{=0}}$ net invest

$\cdot$ $(\underline{w}^N)'\underline{R}^N = \frac{1}{\sqrt{N}} \frac{1}{\|\underline{v}^N\|_2} (\underline{v}^N)'\left[\underline{\mu}^N + B^N\underline{F} + \underline{\varepsilon}^N\right]$

$= \frac{1}{\sqrt{N}} \frac{1}{\|\underline{v}^N\|_2}\left[(\underline{v}^N)'\underline{\mu}^N + \underbrace{(\underline{v}^N)'B^N}_{=0\ (4.2)(2)}\underline{F} + (\underline{v}^N)'\underline{\varepsilon}^N\right]$

$\cdot \Rightarrow$ (ii) $\mathbb{E}\left[(\underline{w}^N)'\underline{R}^N\right] = \frac{1}{\sqrt{N}} \frac{1}{\|\underline{v}^N\|_2} \cdot (\underline{v}^N)'\underline{\mu}^N$

$= \frac{1}{\sqrt{N}} \cdot \frac{1}{\|\underline{v}^N\|_2} (\underline{v}^N)'\left[\lambda_0^N\underline{e}^N + B\lambda^N + \underline{v}^N\right]$

$\underline{= \frac{1}{\sqrt{N}} \cdot \|v^N\|_2}$

$\cdot$ (iii) $Var\left((\underline{w}^N)'\underline{R}^N\right) = \frac{1}{N} \frac{1}{\|\underline{v}^N\|_2^2} Var\left((\underline{v}^N)'\underline{\mu}^N + (\underline{v}^N)'\underline{\varepsilon}^N\right)$

$= \frac{1}{N} \frac{1}{\|\underline{v}^N\|_2^2} \underbrace{(\underline{v}^N)' \underbrace{Var(\underline{\varepsilon}^N)}_{=\Omega^N} \underline{v}^N}_{\leq \bar{\lambda}\, \underline{v}^N \underline{v}^N}$ max eigenvalue = max stretch

$\underline{\leq \bar{\lambda}/N}$

Assume that (4.2) does not hold true

$\Rightarrow \limsup_{N\to\infty} \frac{1}{N} \|\underline{v}^N\|_2^2 = \delta^2 > 0$

16	05
20	11

then $\limsup_{N\to\infty} \mathbb{E}\left((\underline{w}^N)'\underline{R}^N\right) = \delta > 0$ (ii)

and $\lim_{N\to\infty} Var\left((\underline{w}^N)'\underline{R}^N\right) = 0$ (iii)

$\Rightarrow (\underline{w}^N)_{N\geq 1}$ asymptotic arbitrage opportunity ⚡ $\square$

Remarks: the more risky returns we consider, the better we can characterize the K market risk factors $\underline{F}$ and the "more diversification" we have. This implies that the error term v_i^N disappear in the sense (4.2).

Chapter V: Multiperiod Models

Assume discrete time with a finite time horizon $T<\infty$
$t = 0, 1, 2, \dots, T$.

Goal: Value cash flows $\underline{c} = (c_1, c_2, \dots, c_T)' \in \mathbb{R}^T$

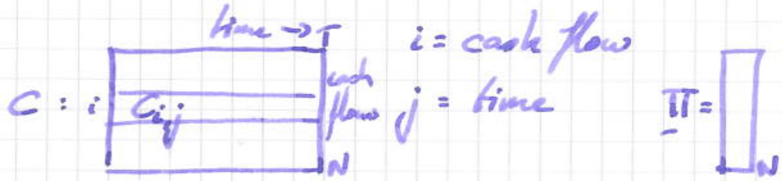
price for $\underline{c}$ at time 0.

5.1. Deterministic cash flows + arbitrage pricing.

Conventions: $\underline{v} \in \mathbb{R}^N$

- $\underline{v} \geq 0 \iff v_i \geq 0 \quad \forall i \in \{1, \dots, N\}$
  $\iff \underline{v} \in \mathbb{R}^N_+ \cup \{\underline{0}\}$
- $\underline{v} > 0 \iff \underline{v} \geq 0$ and $\exists k \in \{1, \dots, N\}$ st $v_k > 0$
  $\iff \underline{v} \in \mathbb{R}^N_+$
- $\underline{v} >> 0 \iff v_i > 0 \quad \forall i \in \{1, \dots, N\}$
  $\iff \underline{v} \in \mathbb{R}^N_{++}$

Def 5.1. A <u>security market</u> is a pair $(\underline{\Pi}, C)$ with $\underline{\Pi} \in \mathbb{R}^N$ and $C \in \mathbb{R}^{N \times T}$



i = cash flow
j = time

Π_i price at time 0.

Def 5.2. A <u>portfolio strategy</u> $\underline{\theta} \in \mathbb{R}^N$
The cashflow generated by $\underline{\theta}$ is
$$C'\underline{\theta} = \left(\sum_{i=1}^N c_{i,1}\theta_i, \dots, \sum_{i=1}^N c_{i,T}\theta_i\right)$$
The price for $\underline{\theta}$ at time 0 is $\underline{\Pi}'\underline{\theta} = \sum_{i=1}^N \Pi_i \theta_i$

Def 5.3 $\underline{\theta} \in \mathbb{R}^N$ is an <u>arbitrage opportunity</u> if one of the following two conditions holds:

a) $\Pi'\underline{\theta} = 0$ and $C'\underline{\theta} > 0$

b) $\Pi'\underline{\theta} < 0$ and $C'\underline{\theta} \geq 0$

Def 5.4 A security market is <u>arbitrage free</u> if it does <u>not</u> contain an arbitrage opportunity.

<u>Lemma 5.5.</u> (Stiemke's Lemma)

Let $A \in \mathbb{R}^{n \times m}$. Then precisely one of the following two statements holds true:

i) $\exists \underline{x} \in \mathbb{R}_{++}^m$ st. $A\underline{x} = 0$ ← orthogonal

ii) $\exists y \in \mathbb{R}^n$ st $y'A > 0$.

use rank of A. □

Prüfungsprotokoll:

Economic Theory of Financial Markets

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Exam 1

Markowitz Problem in full detail:
- General model assumptions
- X = argmax E[u(w1)] as starting point and assumptions on u(x) and R
- Derive the Markowitz Problem using the assumptions above, with E[exp(α*w1)] being the expectation value of a lognormal distribution etc.
- Solve the Markowitz Problem for the case with riskless asset, where only the MV portfolio was asked, not the efficient portfolio

Exam 2

Multiperiod models with deterministic cash flows:
- Security market, cash flows etc.
- 1st FTAP including proof with Stiemkes Lemma, where it is important to understand and be precise on which matrix or vectors span which space etc.
- Duration

Exam 3

Q1: Arbitrage Pricing Theory w/o idiosyncratic risk
- state model assumptions
- explain arbitrage condition
- what can you do with that? (theorem, explain)
- proof in details

Q2: Arbitrage Pricing Theory with idiosyncratic risk
- explain differences with the previous model
- explain the main result
- what is the key assumption and how does it enter the proof? (*eigenvalues of the covariance matrix describing the idiosyncratic risk are uniformly bounded in N*)

Exam 4

Thema: Einleitungsbeispiel

Beispiel aus der Einleitung rekonstruieren, 2 Phasen, deterministisch. r vorgegeben.

$C_0 = W_0 – Z$,

$C_1 = W_1 + Z (1+r)$.

Wie bestimmt man das optimale Z? Dazu definiere Nutzenfunktion U (wir haben u exponentiell verwendet):

$U(C_0, C_1) = u(C_0) + (1+\delta)^{-1} * u(C_1)$

Maximiere unter Z. Leite also nach Z ab und setze 0. Hier haben wir etwas länger gerechnet, aber nicht zu Ende geführt. Weiter: Wie kann man r bestimmen, falls r nicht vorgegeben ist? Mittels market clearing. Gibt es immer eine eindeutige Lösung für r? Nein, sie existiert nicht immer und muss auch nicht eindeutig sein.

Wahrscheinlich hätte er ein zweites Thema angeschnitten, allerdings war ich gar nicht auf das Kapitel vorbereitet und bin dementsprechend nicht weit gekommen. Atmosphäre sehr angenehm.