B. Wissenschaftliche Mitteilungen

H. Cossette, M. Denuit, J. Dhaene, É. Marceau, Sainte-Foy

Stochastic Approximations of Present Value Functions

1 Introduction

Let V_t be the present value at time 0 of an amount of α_t paid at time t. The stochastic discounted value at time 0 of payments of amount α_t made at times $t = 1, 2, \ldots, n$ is then given by

$$Z_n = V_1 + V_2 + \cdots + V_n \,. \tag{1}$$

Consider for instance an insurance company facing payments of amount α_t at times $t = 1, 2, \ldots, n$; the present value of these n deterministic payments is given by (1).

The V_i's involved in (1) are obviously correlated, so that the convenient independence assumption for the summands in Z_n is not realistic. As a consequence, an exact expression for the cumulative distribution function of Z_n requires the knowledge of the joint distribution of the random vector $(V_1, V_2, \ldots, V_n)$, which is in general not available. Goovaerts, Dhaene and De Schepper (2000) recently proposed to circumvent this problem by approximating Z_n by means of a random variable $\widetilde{Z}_n$ dominating the original Z_n in the convex sense. If we denote by $F_1, F_2, \ldots, F_n$ the respective distribution functions of $V_1, V_2, \ldots, V_n$ involved in (1), $\widetilde{Z}_n$ is given by

$$\widetilde{Z}_n = F_1^{-1}(U) + F_2^{-1}(U) + \cdots + F_n^{-1}(U) \,,$$

where U is a unit uniform random variable and the F_i^{-1}'s are the quantile functions associated to the F_i's. We obviously have that $EZ_n = E\widetilde{Z}_n$ and it can be shown that the inequalities

$$E \max\{Z_n - d, 0\} \leq E \max\{\widetilde{Z}_n - d, 0\} \tag{2}$$

hold for any $d \geq 0$ (that is, Z_n is smaller than $\widetilde{Z}_n$ in the convex order).

Since $\widetilde{Z}_n$ precedes Z_n in the convex sense, the approximation $\widetilde{Z}_n$ is considered as less favorable by all the risk-averse decision-makers, and the method is thus conservative. Moreover, the cumulative distribution function of $\widetilde{Z}_n$ enjoys an

explicit expression and is particularly easy to handle. On the basis of numerical illustrations performed in a situation where the exact cumulative distribution function of Z_n can be obtained, Goovaerts *et al.* (1999) showed that the cumulative distribution functions of Z_n and $\widetilde{Z}_n$ seem to be rather close.
The problem of estimating the distribution of Z_n has been studied, among others, by Beekman and Fuelling (1991), De Schepper and Goovaerts (1992), Dufresne (1990), Frees (1990), Parker (1994c, 1997), De Schepper, Teunen, Goovaerts (1994) and Vanneste, Goovaerts and Labie (1994). This paper aims to carry on with Goovaerts *et al.*'s (1999) approach by providing lower and upper bounds on Z_n in the stochastic dominance sense, using the method proposed in Denuit, Genest and Marceau (1999). This approach also provides upper and lower bounds on the quantiles of Z_n. In risk management, these quantiles correspond to the Value at Risk at different probability levels. Such bounds cannot be obtained with the aid of the convex approximation $\widetilde{Z}_n$. Indeed, we see from (2) that the stop-loss premium of $\widetilde{Z}_n$ is an upper bound of the stop-loss premium of Z_n; more generally, $E\phi(\widetilde{Z}_n)$ is an upper bound for $E\phi(Z_n)$ for any convex function ϕ. However, there is in general no relation between $P[Z_n \leq z]$ and $P[\widetilde{Z}_n \leq z]$ (since indicator functions are not convex).
Another purpose of this work is to provide several numerical illustrations which enhance the practical interest of our approach. In these illustrations, we will examine the position of the cumulative distribution function corresponding to the convex approximation $\widetilde{Z}_n$ in the admissible region delimitated by the stochastic bounds on Z_n. As a byproduct of our results, the error in the approximation of Z_n by $\widetilde{Z}_n$ can be evaluated (in other words, we get an upper bound for the Kolmogorov distance between Z_n and $\widetilde{Z}_n$).

2 Stochastic bounds on Z_n

In this section, we recall how to build two functions $F_{\min}$ and $F_{\max}$ such that the inequalities

$$F_{\min}(t) \leq P[Z_n \leq t] \leq F_{\max}(t) \quad \text{for all } t \geq 0\,, \tag{3}$$

hold, as well as

$$F_{\min}(t) \leq P[\widetilde{Z}_n \leq t] \leq F_{\max}(t) \quad \text{for all } t \geq 0\,. \tag{4}$$

To this end, we use the following result due to Denuit *et al.* (1999, Proposition 2). Let $F_1, F_2, \ldots, F_n$ be the respective cumulative distribution functions of

$V_1, V_2, \ldots, V_n$. Then, the cumulative distribution function F_{Z_n} of $Z_n = V_1 + V_2 + \cdots + V_n$ is constrained by (3) with

$$F_{\min}(t) = \sup_{(v_1, v_2, \ldots, v_n) \in \Sigma(t)} \max\left\{ \sum_{i=1}^{n} P[V_i < v_i] - (n-1), 0 \right\},$$

and

$$F_{\max}(t) = \inf_{(v_1, v_2, \ldots, v_n) \in \Sigma(t)} \min\left\{ \sum_{i=1}^{n} F_i(v_i), 1 \right\},$$

where

$$\Sigma(t) = \{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \mid v_1 + v_2 + \cdots + v_n = t\}, \quad t \in \mathbb{R}.$$

Note that $F_{\max}$ is a *bona fide* cumulative distribution function, whereas $F_{\min}$ is the left-continuous version of some cumulative distribution function. The bounds in (3) and (4) are the best-possible bounds on Z_n and $\widetilde{Z}_n$ in the sense of stochastic dominance when we know the distribution functions $F_1, F_2, \ldots, F_n$, but no assumption is made on the dependence structure between the V_i's. Equivalently, these bounds hold for all sums (1) with given cumulative distribution functions for $V_1, V_2, \ldots, V_n$.

Closed form expressions for the bounds (3) can in general not be obtained for distributions of the V_i's and one must resort to numerical evaluation. For more details, see Denuit *et al.* (1999).

Now, assume we have at our disposal some partial knowledge of the dependence existing between the V_i's, namely that there exists a multivariate cumulative distribution function G satisfying

$$\begin{aligned} &G(v_1, v_2, \ldots, v_n) \le P[V_1 \le v_1, V_2 \le v_2, \ldots, V_n \le v_n] \\ &\qquad \text{for all } v_1, v_2, \ldots, v_n \in \mathbb{R}, \end{aligned} \tag{5}$$

and a joint decumulative distribution function $\overline{H}$ such that

$$\begin{aligned} &P[V_1 > v_1, V_2 > v_2, \ldots, V_n > v_n] \ge \overline{H}(v_1, v_2, \ldots, v_n) \\ &\quad \text{for all } v_1, v_2, \ldots, v_n \in \mathbb{R}. \end{aligned} \tag{6}$$

From Denuit *et al.* (1999, Proposition 5), the inequalities

$$\begin{aligned} &\sup_{(x_1, x_2, \ldots, x_n) \in \Sigma(t)} G(x_1, x_2, \ldots, x_n) \\ &\le F_{Z_n}(t) \\ &\le 1 - \sup_{(x_1, x_2, \ldots, x_n) \in \Sigma(t)} \overline{H}(x_1, x_2, \ldots, x_n), \end{aligned} \tag{7}$$

hold for all $t \in \mathbb{R}$. The bounds in (7) are obviously more accurate than those in (3).
In the literature, several notions of positive dependence have been introduced in order to express the fact that large values of one of the components of a random vector tend to be associated with large values of the others. In our context, one intuitively feels that in most situations the V_i's mainly "move together" (i. e. a large value of V_i is usually followed by a large value of V_{i+1}). For the numerical illustrations in this paper, we will assume that (5) and (6) are satisfied with

$$G(v_1, v_2, \dots, v_n) = \prod_{i=1}^{n} F_i(v_i)$$

and

$$\overline{H}(v_1, v_2, \dots, v_n) = \prod_{i=1}^{n} (1 - F_i(v_i)) \,.$$

In such a case, the V_i's are said to be Positively Orthant Dependent (POD, in short). POD comes thus down to assume that the probability that all the V_i's assume "small" values (i. e. $V_i \leq v_i$, $i = 1, 2, \dots, n$) is larger than the corresponding probability under the assumption that the V_i's are mutually independent. The interpretation for $\overline{H}$ is similar by substituting "large" for "small". For more details, see, e. g., Szekli (1995, pp. 144–145).

3 Applications

3.1 Stochastic annuities

Let δ_s be the force of interest at time s and let Y_t denote the force of interest accumulation function at time t, i. e.

$$Y_t = \int_{s=0}^{t} \delta_s \, ds \,.$$

The random present value at time 0 of a payment of 1 monetary unit at time t is given by $\exp(-Y_t)$, $t \geq 0$.
As noticed by Parker (1994b), there are mainly two possible approaches to model the interest randomness, namely the modeling of Y_t and the modeling of δ_s. In the first approach, we could let Y_t be the sum of a deterministic drift of slope δ and a perturbation modeled by a Wiener process, i. e.

$$Y_t = \delta t + \sigma W_t, \quad t \in \mathbb{R}^+, \tag{8}$$

where σ is a non-negative constant and $\{W_t, \ \ t \in \mathbb{R}^+\}$ is a standardized Brownian motion. In such a case, V_t is log-normally distributed with parameters $-\delta t$ and $\sigma^2 t$. This corresponds to the approach adopted by Goovaerts *et al.* (1999) who considered a discounted cash flow Z_n of the form

$$Z_n = \sum_{i=1}^{n} \exp(-\delta i - X_i)\,,$$

where the X_i's are assumed to be normally distributed with mean 0 and variance $i\sigma^2$, and δ is the expected force of interest. The convex upper bound $\widetilde{Z}_n$ on Z_n obtained by Goovaerts *et al.* (1999) is

$$\widetilde{Z}_n = \sum_{i=1}^{n} \exp\Big\{-\delta i - \sigma\sqrt{i}\Phi^{-1}(U)\Big\}\,, \tag{9}$$

where Φ is the cumulative distribution function of a standard normal distribution and U is a random variable uniformly distributed on the unit interval $[0, 1]$. The survival function of $\widetilde{Z}_n$ then follows from

$$P[\widetilde{Z}_n > x] = 1 - F_{\tilde{Z}_n}(x) = \Phi(\nu_x)\,,$$

with ν_x the root of the equation

$$\sum_{i=1}^{n} \alpha_i \exp(-\delta i - \sqrt{i}\sigma\nu_x) = x\,.$$

Let us now investigate the accuracy of the bounds (3) and (7) on the distribution function of Z_n in the model (8). In Figure 1, one sees the functions $F_{\min}$ and $F_{\max}$ involved in (3). With in between the approximation $F_{\tilde{Z}_n}$ of the unknown F_{Z_n} for $n = 10$, $\delta = 0.08$ and $\sigma = 0.02$. Figure 3 is the analog for $n = 20$. Comparing the cumulative distribution function of the convex approximation (9) with the stochastic bounds (3), we see from Figures 1 and 3 that (9) lies in the very middle of the admissible region bordered by $F_{\min}$ and $F_{\max}$. This indicates that (9) could be reasonable. In Figures 2 and 4, we further assume that the V_i's are POD and we computed the improved bounds furnished in (3). Only the lower bound got improved. As it is observed in Example 3 of Denuit *et al.* (1999), both upper and lower bounds on the distribution of a sum of random variables got improved when the supports of the random variables are of the form $[a_i, b_i]$ with $-\infty < a_i < b_i < +\infty$. If b_i is equal to $+\infty$ as in Example 1 of Denuit *et al.* (1999), only the lower bound will be improved with the assumption of POD. In our examples, the random variables are lognormally distributed with supports

corresponding to $[0, +\infty)$. If, as in Goovaerts and Dhaene (1999), δ_t is defined by a CIR model, then Y_t will be strictly positive, $V_t = \exp(-Y_t)$ will take values between 0 and 1, and therefore upper and lower bounds on the distribution of Z_t will have been improved.

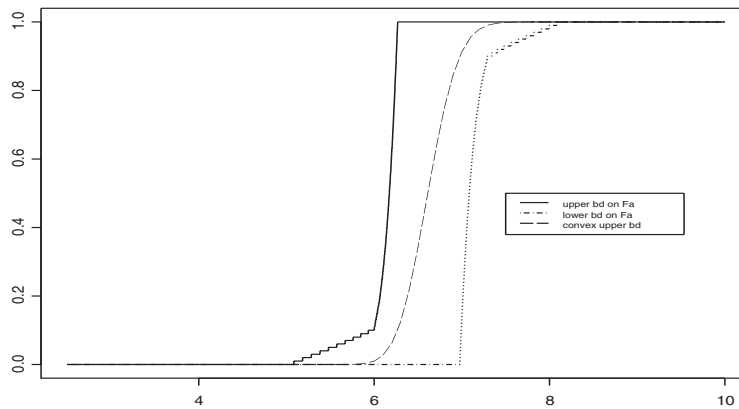


Figure 1: *Graph of the bounds (3) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (8) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

A second approach to model interest randomness is to model δ_s. For instance, the force of interest can be defined by the differential equation

$$d\delta_t = -\alpha(\delta_t - \delta)dt + \sigma dW_t \,, \tag{10}$$

with non-negative constants α and σ, and with initial value $\delta_0 = \delta \geq 0$; $\{\delta_t,\ t \geq 0\}$ is thus an Ornstein-Uhlenbeck process. The force of interest accumulation function $\{Y_t,\ t \geq 0\}$ is therefore a Gaussian process with mean function

$$t \mapsto \mu_t = \delta t + (\delta_0 - \delta)\frac{1 - \exp(-\alpha t)}{\alpha} \,,$$

and autocovariance $(s,t) \mapsto \mathrm{Cov}[Y_s, Y_t] \equiv \omega(s,t)$, where

$$\omega(s,t) = \frac{\sigma^2}{\alpha^2}\min(s,t) + \frac{\sigma^2}{2\alpha^3}\Big\{-2 + 2\exp(-\alpha s) + 2\exp(-\alpha t) \\ - \exp(-\alpha(t-s)) - \exp(-\alpha(t+s))\Big\};$$

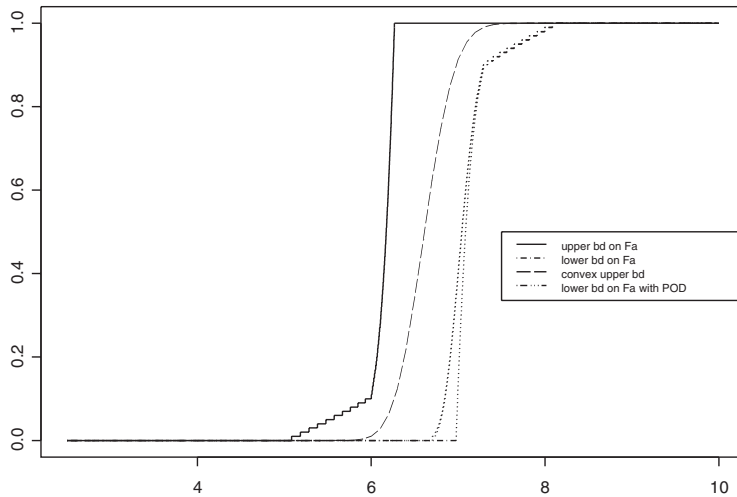


Figure 2: *Graph of the bounds (7) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (8) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

see e. g. Parker (1994a, Section 6). Then,

$$Z_n = \sum_{i=1}^{n} \exp(-Y_i)\,,$$

where Y_i is a Normal random variable with mean μ_i and variance $\omega(i,i)$. In such a case, the convex upper bound $\tilde{Z}_n$ follows from Goovaerts *et al.* (1999):

$$\widetilde{Z}_n = \sum_{i=1}^{n} \exp\left\{-\mu_i - \sqrt{\omega(i,i)}\Phi^{-1}(U)\right\},$$

where U is a random variable uniformly distributed on the unit interval $[0,1]$. In Figure 5, you can see the bounds on the cumulative distribution function of Z_{10} in the model (10) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$, together with the cumulative distribution function of $\tilde{Z}_{10}$. Figure 7 is the analog for $n = 20$. The comments inspired from Figures 1 and 3 still apply. In Figures 6 and 8, we assumed that the V_i's were POD. Again, the improvement with POD is moderate.

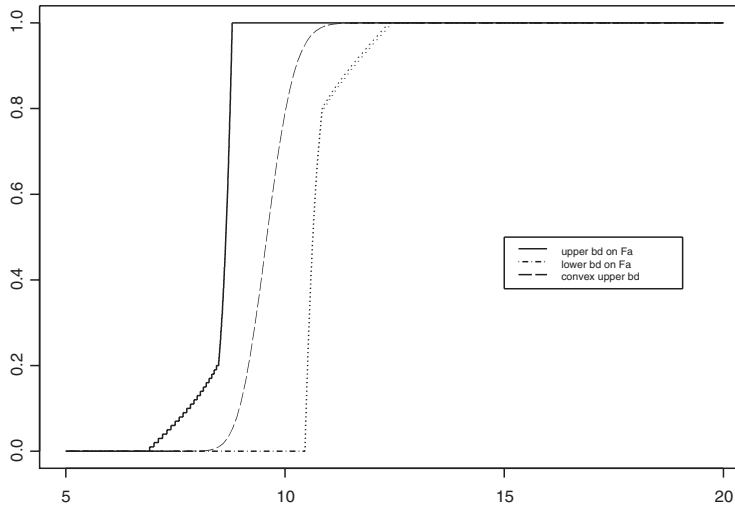


Figure 3: *Graphs of the bounds (3) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (8) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

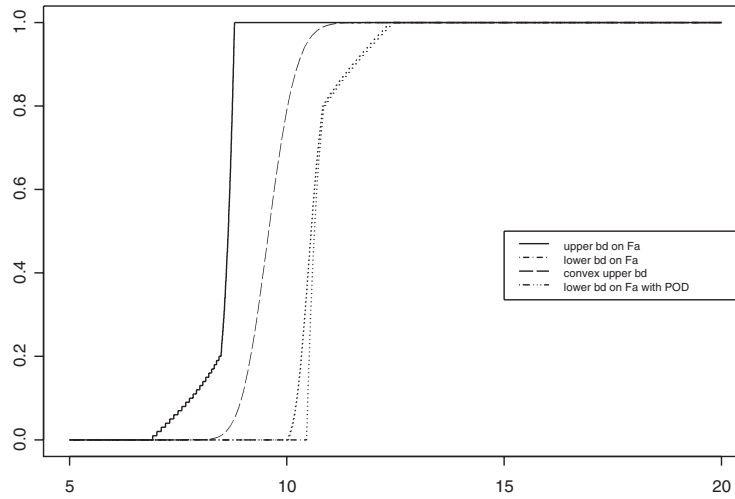


Figure 4: *Graphs of the bounds (7) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (8) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

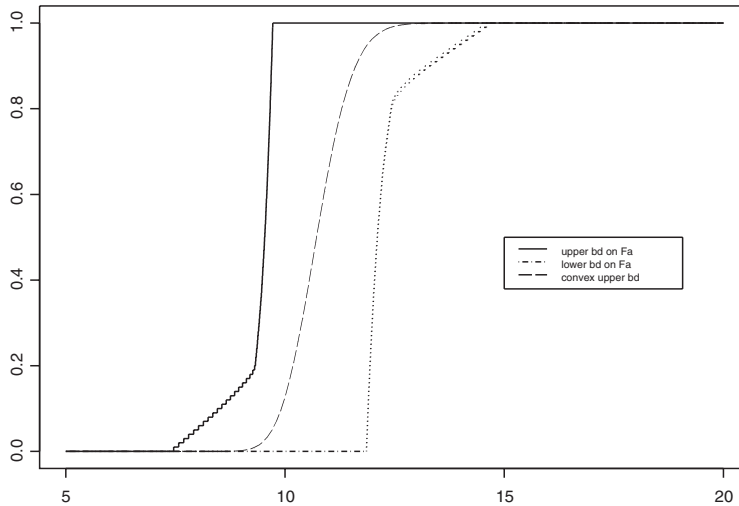


Figure 5: *Graphs of the bounds (3) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (10) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

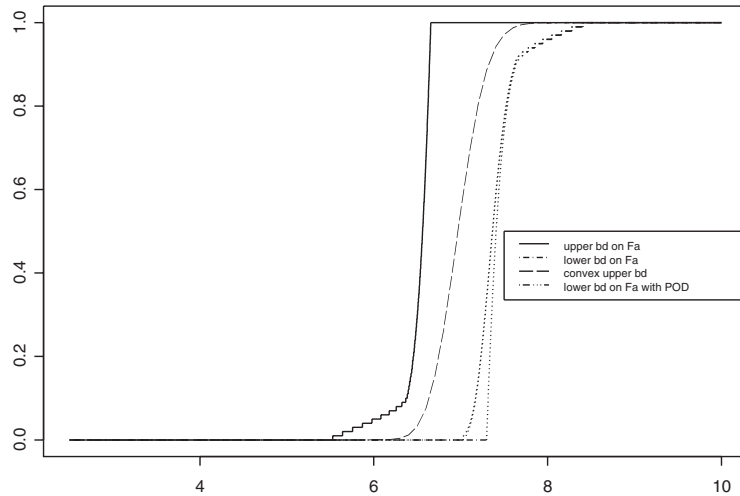


Figure 6: *Graphs of the bounds (7) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (10) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

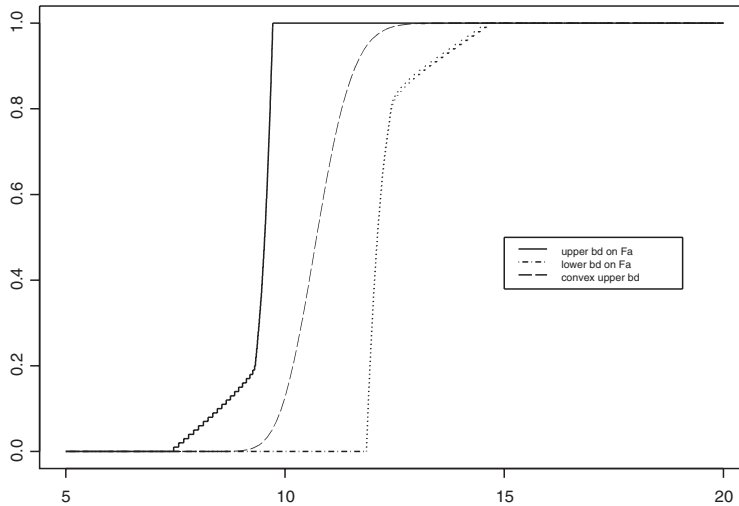


Figure 7: *Graphs of the bounds (3) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (10) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

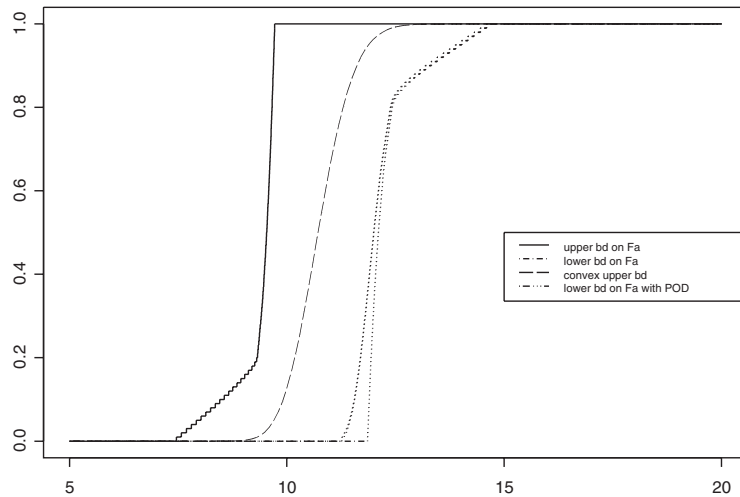


Figure 8: *Graphs of the bounds (7) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (10) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

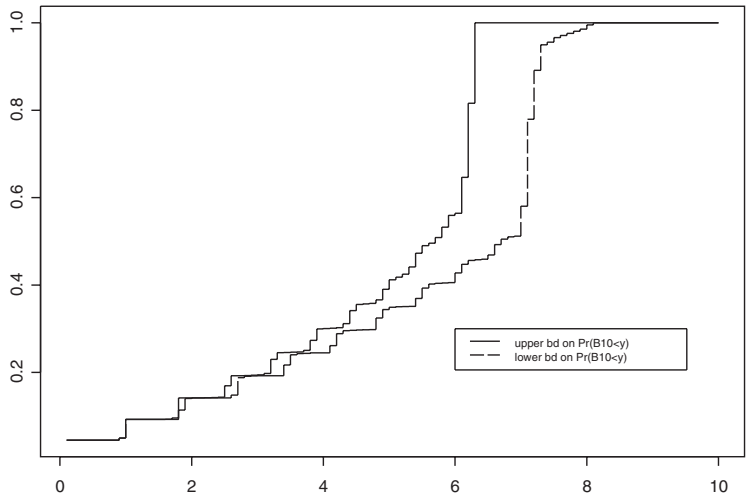


Figure 9: *Bounds on* $P[a^{\circ}_{x;\overline{10|}} \leq y]$ *for* $x = 45$ *and (8) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

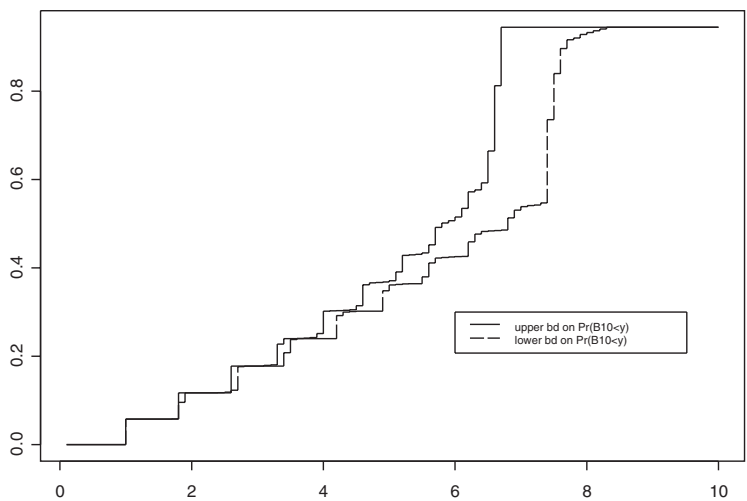


Figure 10: *Bounds on* $P[a^{\circ}_{x;\overline{10|}} \leq y]$ *for* $x = 45$ *and (10) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

3.2 *Life insurance*

Consider a temporary life annuity issued to an individual aged x with curtate-future-lifetime K and denote $P[k < K \leq k+1] = {}_{k|}q_x$ and $P[K > n] = {}_n p_x$. We assume that K is independent of the random discount factors $V_1, V_2, V_3, \ldots$. The net single premium relating to this contract is given by

$$a_{x;\overline{n|}} = E[a^{\circ}_{x;\overline{n|}}]\,,$$

with

$$a^{\circ}_{x;\overline{n|}} = \begin{cases} 0 & \text{if } K = 0\,, \\ Z_K & \text{if } K = 1, \ldots, n-1\,, \\ Z_n & \text{if } K \geq n\,, \end{cases}$$

where Z is defined as in (1). By conditioning on K, the net single premium relating to such a contract is

$$a_{x;\overline{n|}} = \sum_{k=1}^{n-1} E[Z_k]_{k|}q_x + E[Z_n]_n p_x\,.$$

The cumulative distribution function of $a^{\circ}_{x;\overline{n|}}$ is also obtained by conditioning on K:

$$P[a^{\circ}_{x;\overline{n|}} \leq y] = q_x + \sum_{k=1}^{n-1} P[Z_k \leq y]_{k|}q_x + P[Z_n \leq y]_n p_x\,.$$

No explicit expression exists for $P[a^{\circ}_{x;\overline{n|}} \leq y]$, but we use the approach developed above allows us to find stochastic dominance bounds on $a^{\circ}_{x;\overline{n|}}$. In Figure 9, we depicted the graph of the bounds on $P[a^{\circ}_{x;\overline{n|}} \leq y]$ for an individual aged 45 in the model (8) with $\delta = 0.08$ and $\sigma = 0.02$. Figure 10 is the analog in model (10) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$. For these numerical illustrations, we used the standard mortality table (Makeham model) given in Bowers *et al.* (1996). The bounds in Figures 9 and 10 give a good idea of the danger inherent to the stochastic interest rate combined with the stochastic mortality. Let us mention that the convex approximation of Goovaerts *et al.* (1999) also applies in this situation.

Acknowledgements

Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada and the "Chaire en Assurance l'Industrielle-Alliance".

References

[1] Beekman, J. A., and C. P. Fuelling (1991). Extra randomness in certain annuity models. *Insurance: Mathematics & Economics* **10**, 275–287.

[2] Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, O. A. and C. J. Nesbitt (1996). *Actuarial Mathematics*. Society of Actuaries, Itasca, Illinois.

[3] Denuit, M., Genest, C. and E. Marceau (1999). Stochastic bounds on sums of dependent risks. *Insurance: Mathematics & Economics* **25**, 85–104.

[4] De Schepper, A., and M. J. Goovaerts (1992). Some further results on annuities certain with random interest. *Insurance: Mathematics & Economics* **11**, 283–290.

[5] De Schepper, A., Teunen, M., and M. J. Goovaerts (1994). An analytical inversion of a Laplace transform related to annuity certain. *Insurance: Mathematics & Economics* **14**, 33–37.

[6] Dufresne, D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal* 33–79.

[7] Frees, E. W. (1990). Stochastic life contingencies with solvency considerations. *Transactions of the Society of Actuaries* **XLII**, 91–148.

[8] Goovaerts, M. J., and Dhaene, J. (1999). Supermodular ordering and stochastic annuities. *Insurance: Mathematics & Economics* **24**, 281–290.

[9] Goovaerts, M., Dhaene, J., and De Schepper, A. (2000). Stochastic upper bounds for present value functions. *Journal of Risk and Insurance* **67**, 1–14.

[10] Parker, G. (1994a). Limiting distribution of the present value of a portfolio. *ASTIN Bulletin* **24**, 47–60.

[11] Parker, G. (1994b). Two stochastic approaches for discounting actuarial functions. *ASTIN Bulletin* **24**, 167–181.

[12] Parker, G. (1994c). Stochastic analysis of a portfolio of endowment policies. *Scandinavian Actuarial Journal*, 119–130.

[13] Szekli, R. (1995). *Stochastic Ordering and Dependence in Applied Probability*. Lecture Notes in Statistics **97**. Springer Verlag. Berlin.

[14] Vanneste, M., Goovaerts, M. J., and E. Labie (1994). The distribution of annuities. *Insurance: Mathematics & Economics* **15**, 37–48.

Michel Denuit
Institut de Statistique
Université Catholique de Louvain
Voie du Roman Pays, 20
B-1348 Louvain-la-Neuve
Belgium
E-mail adress: denuit@stat.ucl.ac.be

Abstract

The aim of this paper is to apply the method proposed by Denuit, Genest and Marceau (1999) for deriving stochastic upper and lower bounds on the present value of a sequence of cash flows, where the discounting is performed under a given stochastic return process. The convex approximation provided by Goovaerts, Dhaene and De Schepper (2000) and Goovaerts and Dhaene (1999) is then compared to these stochastic bounds. On the basis of several numerical examples, it will be seen that the convex approximation seems reasonable.

Résumé

Le but de cet article est d'appliquer la méthode proposée par Denuit, Genest et Marceau (1999) afin d'obtenir des bornes supérieures ou inférieures au sens de la dominance stochastique pour la valeur actuelle d'une série de flux financiers lorsque le taux d'intérêt obéit à un processus stochastique donné. L'approximation au sens de l'ordre convexe proposée par Goovaerts, Dhaene et De Schepper (2000) et Goovaerts et Dhaene (1999) est ensuite comparée aux bornes évoquées plus haut. Sur base de plusieurs exemples numériques, l'approximation convexe semble raisonnable.

Zusammenfassung

Ziel dieses Artikels ist die Anwendung einer Methode, die von Denuit, Genest und Marceau (1999) vorgeschlagen wurde, um stochastische obere und untere Schranken für den Barwert einer Reihe von Cash Flows zu erhalten, wobei die Diskontierung gemäss eines gegebenen stochastischen Prozesses durchgeführt wird. Die von Goovaerts, Dhaene und De Schepper (2000) sowie Goovaerts und Dhaene (1999) vorgeschlagene konvexe Approximation wird anschliessend mit diesen stochastischen Schranken verglichen. Anhand mehrerer numerischer Beispiele wird gezeigt, dass die konvexe Approximation vernünftig aussieht.

J. F. WALHIN, Louvain-la-Neuve

Some comments on two approximations used for the pricing of reinstatements

1 Introduction

Excess of loss reinsurance is widely used when it is the concern of the ceding company to reduce the cost of individual major losses. If D is the retention of the insurance company, then the reinsurer is liable for

$$R_i = \max(0, X_i - D)\,, \tag{1}$$

where X_i denotes the amount of the ith claim.
Often the reinsurer limits his liability to a level L. $[D, D+L]$ is called the layer. (1) becomes

$$R_i = \min(L, \max(X_i - D, 0))\,. \tag{2}$$

If there are N claims during the year, the aggregate claims distribution of the reinsurer is

$$S_R = R_1 + \cdots + R_N\,. \tag{3}$$

Unfortunately for the ceding company, the reinsurer does not always offer unlimited capacity. So he will try to limit his aggregate liability. A common way for doing this is to use reinstatements.
It is said that the reinsurer offers k reinstatements if he limits his yearly liability to $k+1$ times the layer L. Sometimes the reinstatements are free. Often they are paid. This means that when a claim affects the layer, the ceding company is obliged to pay a reinstatement premium in order to reinstate the layer.
This reinstatement premium is calculated pro rata capita, i. e. it considers the fraction of the layer used by the potential claims leading to the jth reinstatement premium:

$$\frac{1}{L}\min(L, \max(0, S_R - (j-1)L))\,.$$

This fraction is multiplied by a certain price, or percentage (c_j) of the initial premium P:

$$\frac{Pc_j}{L}\min(L, \max(0, S_R - (j-1)L))\,.$$

The latter formula gives the reinstatement premium to be collected in order to reinstate the layer for the jth time. Note that a single claim R_i may lead to a payment with respect to reinstatements j and $j+1$. Indeed, assume a cover $20\,xs\,10$ (i. e. $D = 10$ and $L = 20$) with 2 reinstatements, the first one payable at $c_1 = 100\%$ and the second one payable at $c_2 = 200\%$. The initial premium is $P = 10$.

Assume a first claim $X_1 = 25$, i. e. $R_1 = 15$. The reinstatement premium is $\frac{15}{20} \times 100\% \times 10 = 7.5$.

Assume a second claim $X_2 = 23$, i. e. $R_2 = 13$. R_2 is split in two parts, one for the first reinstatement, one for the second reinstatement: $R_2 = 5 + 8$. The total reinstatement premium is $\frac{5}{20} \times 100\% \times 10 + \frac{8}{20} \times 200\% \times 10 = 2.5 + 8 = 10.5$.

Nevertheless we do not need to write the formulae giving the reinstatement premium regarding claim i. Indeed the reinstatement premiums are also found by:

$$\begin{aligned} S_R &= 28 \\ \text{First reinstatement} &= \frac{10 \times 100\%}{20} \min(20, \max(0, 28 - 0)) \\ &= 10 \\ \text{Second reinstatement} &= \frac{10 \times 200\%}{20} \min(20, \max(0, 28 - 20)) \\ &= 8 \end{aligned}$$

So we observe that the formula does not depend on the individual R_i's but only on the aggregate S_R.

However this is not the case when there are pro rata temporis reinstatements. In fact, apart from the pro rata capita calculations, the reinstatement premium is sometimes calculated pro rata temporis as well. This means that we take into account the time remaining until the next reinsurance renewal $(1 - T_i)$ where T_i is the time elapsed until the ith claim.

Recall our numerical example and assume that the first claim arises at time $T_1 = 0.2$ whereas the second claim arises at time $T_2 = 0.6$. We assume that the time between two renewals is 1.

The reinstatement premium for the first claim writes $\frac{15}{20} \times 100\% \times 10 \times (1 - 0.2) = 6$.

The reinstatement premium for the second claim writes $\frac{5}{20} \times 100\% \times 10 \times (1 - 0.6) + \frac{8}{20} \times 200\% \times 10 \times (1 - 0.6) = 1 + 3.2 = 4.2$.

In this case we have to write the formulae giving the reinstatement premium with respect to claim i. We shall come back to this point in section 4.

It is clear that pricing such specific clauses is not trivial. The case of pro rata capita only is treated in Sundt (1991) and in Walhin and Paris (2001). In section 2 we give the exact formula for the pricing of a pro rata capita only clause. In section 3 we study an approximation called the rate on line method. To the knowledge of the author, the case of pro rata temporis reinstatements is not treated in the literature. In section 4 we give the exact formula when there is only one reinstatement, which represents the most common case. We make some comments on a trivial approximate formula in section 5. Some extensions, including the case of an aggregate deductible and more than one reinstatement are discussed in section 6. Section 7 gives a conclusion.

Throughout the paper we will use a numerical example, based on the following assumptions:
The X_i, N and T are independent.
N: Poisson with mean λ.
X: truncated Pareto with parameters A, B and α:

$$F_X(x) = 0\,, \qquad x \leq A\,,$$

$$F_X(x) = \frac{A^{-\alpha} - x^{-\alpha}}{A^{-\alpha} - B^{-\alpha}}\,, \qquad A < x \leq B\,,$$

$$F_X(x) = 1\,, \quad x > B\,,$$

where $F_X(x)$ denotes the cumulative density function of the random variable X.
T: Beta distributed with parameters a and b

$$f_T(t) = \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1}\,, \qquad 0 \leq t \leq 1\,,\ a > 0\,,\ b > 0\,,$$

where $f_T(t)$ denotes the density function of the random variable T and $B(a,b)$ is the Beta function:

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt\,.$$

Note that in the particular case where $a = b = 1$ the Beta distribution degenerates to the Uniform distribution.

It is important to mention that within a homogeneous Poisson process, the interarrival times are exponential distributed which implies that the random variable T should be Gamma distributed.

However we want to work with a process with possible varying intensity during the year. Furthermore we are only interested in

- a technical fit with respect to the number of claims per year.
- the arrival times of the claims within the year.

These concepts are easily modelled by

- a Poisson distribution (but we might also take a Negative Binomial distribution e. g.). Note that for numerical reasons we will restrict ourselves, later, to a truncated Poisson distribution.
- a Beta distribution which reproduces easily different patterns of arrival times within a given period.

So it should be clear that we work with technical fits of a Poisson distribution and a Beta distribution.

We will take $\lambda = 1$ (unless otherwise stated), $A = 20$, $B = 50$, $\alpha = 1.5$, $a = 0.5, 1, 5$ and $b = 0.5, 1, 5$.

We will study 4 reinsurance excess of loss covers:

- 30 xs 20,
- 10 xs 20,
- 10 xs 30,
- 10 xs 40.

As we will use convolutions or the algorithm of Panjer (1981), we need a discretization of the random variable X. The discrete distribution X_{dis} has been obtained by the minimization of the Kolmogorov distance between the exact distribution and the approximate one (see Walhin and Paris (1998)). The minimization has been handled with the following constraints:

- X_{dis} has the following support: $20, 21, \ldots, 49, 50$,
- $F_{X_{dis}}(B) = 1$,
- $f_{X_{dis}}(x) \geq 0 \quad \forall x = 20, \ldots, 50$,
- $\mathbb{E}X_{dis} = \mathbb{E}X$,
- $\mathbb{V}arX_{dis} = \mathbb{V}arX$,
- $\mathbb{E}\min(10, \max(0, X_{dis} - 20)) = \mathbb{E}\min(10, \max(0, X - 20))$,
- $\mathbb{E}\min(10, \max(0, X_{dis} - 30)) = \mathbb{E}\min(10, \max(0, X - 30))$,

in order to keep the first two moments as well as the expectations on each layer we will study. The expectation on the layer 10 xs 40 is automatically kept by linear combination of the defined constraints.
The Kolmogorov distance between the exact and the discretized distribution is 0.0472.

2 Pro rata capita: exact formula

An initial premium P is paid. Reinstatement premiums, functions of P, might be paid. This random part of the premium income is

$$P \sum_{j=1}^{k} \frac{c_j}{L} \min(L, \max(0, S_R - (j-1)L)) \,.$$

The reinsurer covers $(k+1)$ times the layer:

$$\min(S_R, (k+1)L) \,.$$

The premium P is the solution to the problem of equating the mean premium and the mean aggregate claims:

$$\begin{aligned} &\mathbb{E}\left[P\left(1 + \sum_{j=1}^{k} c_j \min\left(1, \max\left(0, \frac{S_R - (j-1)L}{L}\right)\right)\right)\right] \\ &\quad = \mathbb{E}\min(S_R, (k+1)L) \,. \end{aligned}$$

This immediately gives

$$P = \frac{\mathbb{E}\min(S_R, (k+1)L)}{1 + \sum_{j=1}^{k} \frac{c_j}{L} \mathbb{E}\min(L, \max(0, S_R - (j-1)L))} \,.$$

This is the premium given by the expected value premium principle studied in Sundt (1991). Sundt(1991) also studies the standard deviation principle while Walhin and Paris (2001) study the PH transform principle.

In this paper we will only concentrate on the expected value premium principle.

The following table gives the premium for the 30 xs 20 cover with uniform price $c_j \equiv c$ of the reinstatement.

c/k	0	1	2	3	4
0%	8.75	9.48	9.51	9.52	9.52
50%		8.28	8.21	8.21	8.21
100%		7.34	7.23	7.22	7.22
150%		6.59	6.45	6.45	6.45

Table 1: 30 xs 20: pure premium

3 The rate on line method

The rate on line is defined as

$$ROL = \frac{\mathbb{E}N\mathbb{E}R}{L}.$$

It is the premium of an unlimited free reinstatements treaty covering the layer $[D, D+L]$ divided by the lenght of the layer L.

Let us assume that we do not know the distribution of S_R. Instead, we know the rate on line.

The rate on line method assumes that there are only total losses, i. e. losses hitting the layer completely. For such losses, the frequency is the ROL.

Let us define two new random variables:

R' = L with probability 1,

N' has the same distribution as N but with mean ROL.

It is then easy to show that the formula for P becomes

$$P = L\frac{\sum_{i=0}^{k} \mathbb{P}(N' > i)}{1 + \sum_{i=1}^{k} c_i \mathbb{P}(N' > i-1)}.$$

This formula is obviously easier to evaluate than the exact one.

Our numerical example gives the following approximate premiums:

c/k	0	1	2	3	4
0%	8.15	9.38	9.50	9.51	9.52
50%		8.26	8.22	8.21	8.21
100%		7.37	7.24	7.22	7.22
150%		6.66	6.47	6.45	6.45

Table 2: 30 xs 20: approximate pure premium

We note that, in case of free reinstatements, the exact premium is always higher than the approximate one. This is obviously not a safe approximation and it can be shown on numerical examples that the error may be larger than 10%.
We will now prove that the conjecture mentioned above is always true.

Proposition 1 *Let N be such that its probability generating function $\psi_N(z;\theta) = W(\theta(z-1))$ where W is a function independent of the parameter θ.*
Then the approximate premium is always lower than the exact premium if the reinstatements are free.

Let us define the random variable $R^{''}$:

x	$\mathbb{P}[R'' = x]$
0	$1 - \dfrac{\mathbb{E}R}{L}$
L	$\dfrac{\mathbb{E}R}{L}$

Let us denote by $\psi_S(z) = \mathbb{E}[z^S]$ the probability generating function of S.
The probability generating function of $S^{''} = R_1^{''} + \cdots + R_N^{''}$ is given by

$$\psi_{S_{R''}}(z) = \psi_N(\psi_{R''}(z);\theta)\,.$$

By a theorem of Panjer and Willmot (1984) we can write

$$\begin{aligned}\psi_{S_{R''}}(z) &= \psi_N\left(1 - \frac{\mathbb{E}R}{L} + \frac{\mathbb{E}R}{L}\psi_{R'}(z);\theta\right)\\ &= \psi_N\left(\psi_{R'}(z);\frac{\mathbb{E}R}{L}\theta\right)\end{aligned}$$

and so we have proved that

$$\psi_{S_{R''}}(z) = \psi_{S_{R'}}(z)$$

which shows that $S_{R''} = S_{R'}$.
Among the class of risks Y with $\mathbb{E}Y = \mathbb{E}R$ and $\mathbb{P}[0 \leq Y \leq L] = 1$, R'' is the riskiest with respect to the stop-loss order (see Goovaerts et al. (1990), p. 27). Hence

$$SL(R'', t) \geq SL(R, t)\,, \quad \forall t\,,$$

where $SL(R, t)$ denotes the stop loss premium for a risk R with retention t:

$$SL(R, t) = \int_t^\infty (x - t)\, dF_R(x)\,.$$

By Bühlmann et al. (1977), the stop loss order is preserved under compounding:

$$SL(S_{R''}, t) \geq SL(S_R, t)\,, \quad \forall t\,.$$

With free reinstatements we have

$$\begin{aligned} P &= \mathbb{E}S_R - SL(S_R, (t+1)L), \\ P' &= \mathbb{E}S'_R - SL(S'_R, (t+1)L)\,. \end{aligned}$$

Combined with the fact that $\mathbb{E}S_R = \mathbb{E}S_{R'}$ we find

$$P' \leq P\,. \qquad \blacksquare$$

In particular the proposition is interesting when N is Poisson ($W(x) = e^x$ with $\theta = \lambda$) or Negative Binomial ($W(x) = (1-x)^{-r}$ with $\theta = \beta$).
The following tables give the fraction $\frac{\text{approximate premium}}{\text{exact premium}}$ for the layers 30 xs 20, 10 xs 20, 10 xs 30 and 10 xs 40 with uniform price (c) of the reinstatement.

c/k	0	1	2	3	4
0%	0.93174	0.98889	0.99898	0.99993	0.99999
50%		0.99757	1.00049	1.00007	1.00000
100%		1.00438	1.00165	1.00017	1.00001
150%		1.00988	1.00256	1.00026	1.00001

Table 3: 30 xs 20: approximate / exact premium

c/k	0	1	2	3	4
0%	0.98554	0.99379	0.99837	0.99970	0.99996
50%		0.99659	0.99983	1.00010	1.00003
100%		0.99849	1.00073	1.00034	1.00008
150%		0.99986	1.00135	1.00050	1.00011

Table 4: 10 xs 20: approximate / exact premium

c/k	0	1	2	3	4
0%	0.99100	0.99834	0.99983	0.99999	1.00000
50%		0.99924	1.00001	1.00001	1.00000
100%		0.99997	1.00016	1.00002	1.00000
150%		1.00058	1.00028	1.00003	1.00000

Table 5: 10 xs 30: approximate / exact premium

c/k	0	1	2	3	4
0%	0.98841	0.99957	0.99999	1.00000	1.00000
50%		0.99990	1.00000	1.00000	1.00000
100%		1.00022	1.00002	1.00000	1.00000
150%		1.00052	1.00003	1.00000	1.00000

Table 6: 10 xs 40: approximate / exact premium

We notice that the fraction increases with the price (c) of the reinstatements. This result can be shown analytically.

We also see that the fraction tends to unity with the number of reinstatements (k). This is logical as when $k \to \infty$, the claims distribution (R) is not important. The formula for unlimited uniformly paid reinstatements is

$$P = \frac{\mathbb{E}N\mathbb{E}R}{1 + \frac{c}{L}\mathbb{E}N\mathbb{E}R},$$

where we only use $\mathbb{E}R$.

Now let us assume that there is an aggregate deductible (AD), i. e. the reinsurer is liable for the aggregate claims in excess of AD. It is not difficult to extend the formulae and the following example shows that the approximation is not controlled and may give very bad results.

Let us assume an aggregate deductible (AD=15) for the cover 10 xs 20. We find

c/k	0	1	2	3	4
0%	1.23881	1.27606	1.28599	1.28773	1.28796
50%		1.26636	1.27298	1.27403	1.27415
100%		1.25740	1.26113	1.26157	1.26160
150%		1.24909	1.25029	1.25021	1.25015

Table 7: 10 xs 20 with $AD = 15$: approximate / exact premium

The results are bad and this is not surprising as we know from Bühlmann et al. (1977) that

$$SL(S'_R, AD) > SL(S_R, AD)\,,$$

i. e. the unlimited free reinstatement premium is higher for the approximate case.

Note that Rytgaard (1996) studied the pricing of pro rata capita reinstatements by discretizing the claims distribution with the so-called mass dispersal method. This method is also known as local moment matching with one moment (see Gerber (1982)). An interesting point is that with the crudest discretization, i. e. with only two points, one falls on the rate on line method. Sensitivity analyses on the discretization step are provided by Rytgaard (1996).

4 Pro rata temporis: exact formula

The pro rata temporis clause is seldom used. However when it is used, it essentially is for catastrophe reinsurance where typically only one reinstatement is offered by the reinsurer. In this section we will only consider this particular case.

We now have to take into account the time remaining until the renewal after each claim. Thus we introduce a new random variable T, the remaining time. We also have to study the random variable Y_i, part of the ith claim leading to a reinstatement premium. The Y_i are defined as

$$Y_1 = R_1\,,$$

$$Y_i = \min\left[\max\left(0, L - \sum_{l=1}^{(i-1)} Y_l\right), R_i\right], \quad i \geq 2\,.$$

Let us recall that the R_i are assumed to be independent. So we have

$$f_{R_1,\ldots,R_r}(x_1,\ldots,x_r) = f_{R_1}(x_1) \times \cdots \times f_{R_r}(x_r)\,.$$

The Y_i are associated with the order statistic of T: $T_{(1:r)} \leq \cdots \leq T_{(r:r)}$.
From now $i \in \{1,\ldots,r\}$ will also denote the index of the ith order statistic: $T_{(i:r)}$ when we consider r claims.
The exact premium for an excess of loss reinsurance with pro rata capita and pro rata temporis reinstatements is:

$$P = \frac{\mathbb{E}\min(S_R, 2L)}{1 + \sum\limits_{n=1}^{r} \mathbb{P}(N=n) \sum\limits_{i=1}^{n} \frac{c}{L}\mathbb{E}(1 - T_{(i:n)})\mathbb{E}Y_i}\,. \tag{4}$$

Unfortunately, evaluating $\mathbb{E}Y_i$ even for i small is very time consuming. Indeed, the random variables $Y_1,\ldots,Y_r$ are highly correlated. So, evaluating $\mathbb{E}Y_i$ requires an i-multiple integral (or sum in the discrete case).
The expectations $\mathbb{E}T_{(i:r)}$ are not too complicated. Indeed, it is not difficult to show that the distribution of the i-th order statistic of T is given by

$$F_{T_{(i:r)}}(u) = \sum_{l=i}^{r} \frac{r!}{l!(r-l)!} F_T^l(u)(1 - F_T(u))^{r-l}\,.$$

Then we immediately have

$$\mathbb{E}(T_{i:r}) = \int_0^\infty \left(1 - \sum_{l=i}^{r} \frac{r!}{l!(r-l)!} F_T^l(u)(1 - F_T(u))^{r-l}\right) du\,.$$

Note that a recursive formula was given by David (1970) in order to compute these expectations:

$$i\mathbb{E}(T_{(i+1:r)}) + (r-i)\mathbb{E}(T_{(i:r)}) = r\mathbb{E}(T_{(i:r-1)})\,.$$

For the practical evaluation of a premium, we have to assume a maximum number of claims r. This is not a problem as these treaties, if proposed with reinstatements pro rata temporis, are essentially used for natural perils for which we do not expect a high frequency.
For our numerical example, we will use a truncated Poisson random variable. We choose to truncate at $r = 4$. The truncation is such that $\mathbb{P}(N \geq 4)$ accumulates at 4.
In order to make different modelizations of the time pattern, we work with a Beta distribution and we let its parameters vary.

We find

i	1	2	3	4
$\mathbb{E}Y_i$	9.52	8.30	5.84	3.42

Table 8: $\mathbb{E}Y_i$ for the 30 xs 20 cover

r/i	1	2	3	4
1	0.5	–	–	–
2	0.58593	0.41407	–	–
3	0.62889	0.5	0.37110	–
4	0.65630	0.54666	0.45333	0.34369

Table 9: $\mathbb{E}(1 - T_{(i:r)})$ for $a = b = 5$

	$a = b = 5$	$a = 5, b = 0.5$	no pro rata temporis
$c = 50\%$	8.80	9.32	8.25
$c = 100\%$	8.23	9.20	7.32
$c = 150\%$	7.73	9.07	6.57

Table 10: Premiums for the 30 xs 20 cover

Obviously the premiums with no pro rata temporis clause are smaller as there will not be any discount for remaining time for the potential reinstatement premiums.

5 Pro rata temporis: a natural approximation

It is clear that a trivial simplification of the formula (4) is to consider that each claim occurs at time $\mathbb{E}T$. Then we assume a mean time remaining equal to $\mathbb{E}(1 - T)$ for each claim. This is a very natural approximation:

$$\begin{aligned} P &= \frac{\mathbb{E}\min(S_R, 2L)}{1 + \mathbb{E}(1-T)\sum_{n=1}^{r}\mathbb{P}(N=n)\sum_{i=1}^{n}\frac{c}{L}\mathbb{E}Y_i} \\ &= \frac{\mathbb{E}\min(S_R, 2L)}{1 + \mathbb{E}(1-T)\frac{c}{L}\mathbb{E}\min(L, \max(0, S_R - L))}\,. \end{aligned}$$

Clearly this formula is far easier to use than the exact formula.

Most of the time, the reinstatements are payable at 100%. We will henceforth work with $c = 100\%$. We have made some comparisons between the exact and the approximate formulae. The tables give the fraction $\frac{\text{exact}}{\text{approximate}}$ for λ, a and b varying as well as the different layers. We find

	$a = b = 5$	$a = 0.5$ $b = 5$	$a = 5$ $b = 0.5$	$a = 0.5$ $b = 0.5$	$a = b = 1$
$\lambda = 0.5$	0.9995	0.9996	0.9996	0.9988	0.9990
$\lambda = 1$	0.9980	0.9988	0.9986	0.9953	0.9961
$\lambda = 2$	0.9929	0.9959	0.9948	0.9836	0.9865

Table 11: 30 xs 20: exact / approximate

	$a = b = 5$	$a = 0.5$ $b = 5$	$a = 5$ $b = 0.5$	$a = 0.5$ $b = 0.5$	$a = b = 1$
$\lambda = 0.5$	0.9968	0.9981	0.9977	0.9925	0.9938
$\lambda = 1$	0.9899	0.9946	0.9920	0.9766	0.9806
$\lambda = 2$	0.9734	0.9867	0.9767	0.9395	0.9497

Table 12: 10 xs 20: exact / approximate

	$a = b = 5$	$a = 0.5$ $b = 5$	$a = 5$ $b = 0.5$	$a = 0.5$ $b = 0.5$	$a = b = 1$
$\lambda = 0.5$	0.9994	0.9996	0.9996	0.9987	0.9989
$\lambda = 1$	0.9980	0.9988	0.9986	0.9955	0.9963
$\lambda = 2$	0.9941	0.9967	0.9956	0.9863	0.9887

Table 13: 10 xs 30: exact / approximate

	$a = b = 5$	$a = 0.5$ $b = 5$	$a = 5$ $b = 0.5$	$a = 0.5$ $b = 0.5$	$a = b = 1$
$\lambda = 0.5$	0.9999	0.9999	0.9999	0.9999	0.9999
$\lambda = 1$	0.9999	0.9999	0.9999	0.9997	0.9998
$\lambda = 2$	0.9996	0.9998	0.9997	0.9992	0.9993

Table 14: 10 xs 40: exact / approximate

It seems clear that the approximate formula is always higher than the exact formula, which is a conservative approximation.

Moreover the error does not seem to be large. This is logical.
In the case of 0 claim, there is no error.
In the case of 1 claim, $\mathbb{E}T = \mathbb{E}T_{(1:1)}$ and so there is no error either.
Thus, the error occurs in case $[N \geq 2]$, which is rather exceptional in catastrophe treaties.
We note that the case $10\,xs\,20$ is the worst, which is logical. Indeed, for that layer, EY_1 will be the largest because it is the layer for which the probability of having a total loss is the highest. If, moreover, $\mathbb{E}(1 - T_{(1:r)})$ is quite different from $\mathbb{E}(1 - T)$ (case $a = b = 0.5$), the approximation is bad. This is almost the worst conceivable case for our approximation.
We will now prove that the above conjecture is always true.

Lemma 1 *For the order statistic $T_{(1:r)}, \ldots, T_{(r:r)}$, we have*

$$\sum_{i=1}^{r} \mathbb{E}T_{(i:r)} = r\mathbb{E}T\,.$$

This result immediately comes from the almost sure equality:

$$T_{(1:r)} + \cdots + T_{(r,r)} = T_1 + \cdots + T_r \qquad \text{a. s.}$$ ■

Proposition 2 *With pro rata temporis reinstatement premiums, the approximate premium is always larger than the exact premium.*

We have to prove that

$$\sum_{n=1}^{r} \mathbb{P}(N = n) \sum_{i=1}^{n} \frac{c}{L} \mathbb{E}(1 - T_{(i:n)}) \mathbb{E}Y_i$$
$$> \mathbb{E}(1 - T) \sum_{n=1}^{r} \mathbb{P}(N = n) \sum_{i=1}^{n} \frac{c}{L} \mathbb{E}Y_i\,.$$

Let us show that the inequality is true for each term of the sum in n:

$$\sum_{i=1}^{n} \mathbb{E}(1 - T_{(i:n)}) \mathbb{E}Y_i > \mathbb{E}(1 - T) \sum_{i=1}^{n} \mathbb{E}Y_i\,.$$

Let γ be such that

$$\mathbb{E}T_{(\gamma:n)} \leq \mathbb{E}T \leq \mathbb{E}T_{(\gamma+1:n)}\,.$$

We have

$$\sum_{i=1}^{\gamma} \mathbb{E}T_{(i:n)}\mathbb{E}Y_i + \sum_{i=\gamma+1}^{n} \mathbb{E}T_{(i:n)}\mathbb{E}Y_i < \sum_{i=1}^{\gamma} \mathbb{E}T\mathbb{E}Y_i + \sum_{i=\gamma+1}^{n} \mathbb{E}T\mathbb{E}Y_i\,,$$

which is equivalent to

$$\sum_{i=\gamma+1}^{n} (\mathbb{E}T_{(i:n)} - \mathbb{E}T)\mathbb{E}Y_i < \sum_{i=1}^{\gamma} (\mathbb{E}T - \mathbb{E}T_{(i:n)})\mathbb{E}Y_i,$$

and the last inequality is always true because

i) $\mathbb{E}Y_1 \geq \mathbb{E}Y_2 \geq \cdots \geq \mathbb{E}Y_n$.

ii) $\sum_{i=\gamma+1}^{n} (\mathbb{E}T_{(i:n)} - \mathbb{E}T) = \sum_{i=1}^{\gamma} (\mathbb{E}T - \mathbb{E}T_{(i:n)})$ because of lemma 1. ■

In the next section we will extend the formulae to the case of an aggregate deductible with pro rata temporis reinstatements and to the case of two reinstatements at different prices.

6 Pro rata temporis: extensions

In the case of an aggregate deductible AD, the random variables R_i are transformed into the random variables Z_i:

$$W_1 = \max(0, R_1 - AD)\,,$$

$$W_i = \max\left(0, R_i - \max\left(0, AD - \sum_{l=1}^{i-1} R_l\right)\right), \quad i \geq 2\,,$$

$$Z_1 = W_1\,,$$

$$Z_i = \min\left[\max\left(0, L - \sum_{l=1}^{(i-1)} Z_l\right), W_i\right], \quad i \geq 2\,,$$

$$P = \frac{\mathbb{E}\min(S_R, 2L)}{1 + \sum_{n=1}^{r} \mathbb{P}(N = n) \sum_{i=1}^{n} \frac{c}{L}\mathbb{E}(1 - T_{(i:n)})\mathbb{E}Z_i}\,.$$

For $\lambda = 1$ and $c = 100\%$, we have evaluated the ratio $\frac{\text{exact}}{\text{approximate}}$ for two aggregate deductibles. We find

	$a=b=5$	$a=0.5$ $b=5$	$a=5$ $b=0.5$	$a=0.5$ $b=0.5$	$a=b=1$
$AD=15$	1.0036	1.0021	1.0025	1.0085	1.0070
$AD=45$	1.0005	1.0003	1.0003	1.0013	1.0010

Table 15: 30 xs 20: exact / approximate

We can see that the approximation is not conservative. This is due to the fact that the $\mathbb{E}Z_i$ are no longer ordered. Note that for some AD, the approximation is conservative.

	$\mathbb{E}Z_1$	$\mathbb{E}Z_2$	$\mathbb{E}Z_3$	$\mathbb{E}Z_4$
$AD=0$	9.52	8.30	5.84	3.42
$AD=15$	1.53	5.06	6.79	6.27
$AD=45$	0	0.09	0.91	2.55

Table 16: $\mathbb{E}Z_i$ for the $30\,xs\,20$ cover

Now let us extend the formulae to the case of two reinstatements payable at prices c_1 and c_2. The case of multiple reinstatements is similar but has no practical interest. This is the reason why we limit ourselves to the case of two reinstatements.
The random variables Y_i are transformed into the random variables Z_i. Note that the random variables Z_i take into account the price of the reinstatement.

$$
\begin{aligned}
Z_1 &= R_1\,,\\
Z_2 &= c_1(L-R_1)+c_2(R_1+R_2-L) \quad \text{if } R_1+R_2\geq L\,,\\
&\qquad c_1R_2 \quad \text{else}\,,\\
Z_k &= c_1R_k \quad \text{if } \sum_{i=1}^{k} R_i \leq L\,,\\
&\qquad c_2\max\left(0, 2L-\sum_{i=1}^{k-1}R_i\right) \quad \text{if } \sum_{i=1}^{k} R_i \geq 2L\,,\\
&\qquad c_1\max\left(0, L-\sum_{i=1}^{k-1}R_i\right)+c_2\left[\min\left(L,\sum_{i=1}^{k-1}R_i\right)+R_k-L\right] \quad \text{else}\,,
\end{aligned}
$$

$$
P=\frac{\mathbb{E}\min(S_R,2L)}{1+\sum\limits_{n=1}^{r}\mathbb{P}(N=n)\sum\limits_{i=1}^{n}\frac{1}{L}\mathbb{E}(1-T_{(i:n)})\mathbb{E}Z_i}\,.
$$

As a numerical example, we take $a = b = 1$ and $\lambda = 1$. We find

c_1	0%	0%	100%	200%	100%	100%	200%	200%
c_2	100%	200%	0%	0%	100%	200%	100%	200%
ratio	0.97	0.97	0.99	1.02	0.99	1.00	1.02	1.02

Table 17: 10 xs 20: exact / approximate

We observe that the approximation cannot be controlled.

7 Conclusion

We have studied throughout this paper two approximations used to price specific clauses for reinsurance treaties.
The rate on line method approximates the premium when there are free or paid reinstatements pro rata capita. We saw that in case of free reinstatements, the approximation is always too low. It is not conservative. As to paid reinstatements, there is no general rule. In conclusion, I wish to remark that this method should not be used with free reinstatements and, in general, the exact method should be preferred since it is not too difficult to set an exact price with the use of recursive techniques like the algorithm of Panjer (1981).
In case of reinstatements which are payable pro rata temporis, we saw that the exact calculation is extremely difficult even with a low frequency of claims. The good news is that the natural simplification is conservative in the sense that it always gives a higher premium than the exact one in the case of one paid reinstatement, which is the most common one. The case of two (or more) reinstatements, which is not common in catastrophe treaties, is not well controlled by the approximation.

Remarks I would like to thank the referee for his carefull reading of the paper and for his constructive remarks.
A fisrt version of this paper has been presented at the XXXIst Astin Colloquium, Porto Cervo, 2000.

References

Bühlmann, H., Gagliardi, B., Gerber, H.U., and Straub, E. (1977). Some Inequalities for Stop-Loss Premiums. *Astin Bulletin*, 9: 75–83.

David, H.A. (1970). *Order Statistics*. John Wiley and Sons, Inc, New York.

Gerber, H.U. (1982). On the Numerical Evaluation of the Distribution of Aggregate Claims and its Stop-Loss Premiums. *Insurance: Mathematics and Economics*, 1: 13–18.

Goovaerts, M.J., Kaas, R., van Heerwaarden, A.E., and Bauwelinckx, T. (1990). *Actuarial Methods*. Insurance Series vol. 3, North-Holland.

Panjer, H.H. (1981). Recursive Evaluation of a Family of Compound Distributions, *Astin Bulletin*, 12: 22–26.

Panjer, H.H. and Willmot, G. (1984). Computational Techniques in Reinsurance Models. *Transactions of the 22nd International Congress of Actuaries, Sydney*, 4: 111–120.

Rytgaard, M. (1996). On Calculating the Risk Premium for an Excess of Loss Cover with an Annual Aggregate Deductible and a Limited Number of Reinstatements. *Proceedings of the XXVIIth Astin Colloquium, Copenhagen 1996*, pages 82–94.

Sundt, B. (1991). On Excess of Loss Reinsurance with Reinstatement. *Bulletin of the Swiss Actuaries*, 1991: 1–15.

Walhin, J.F. and Paris, J. (1998). On the Use of Equispaced Discrete Distributions. *Astin Bulletin*, 28: 241–255.

Walhin, J.F. and Paris, J. (2001). Excess of Loss Reinsurance with Reinstatements: Premium Calculation and Ruin Probability of the Cedent. *BDGVM, 25: 1–12*.

Jean-François Walhin
Secura Belgian Re
Rue Montoyer, 12
B-1000 Bruxelles
Belgique
e-mail: jfw@secura-re.com

Abstract

The aim of this paper is to give some comments on two approximations used to price reinstatements related to excess of loss reinsurance. For the pro rata capita clause, we will study the rate on line method. For the pro rata temporis clause, we will study the use of a trivial approximation. The effect of an aggregate deductible is also looked at.

Résumé

L'objet de cet article est de faire quelques commentaires relatifs à deux approximations utilisées pour la cotation de traités de réassurance excess of loss avec reconstitutions. En ce qui concerne la clause pro rata capita, nous étudions la méthode du rate on line. En ce qui concerne la clause pro rata temporis, nous étudions l'utilisation d'une approximation évidente. L'effet d'une franchise annuelle est également abordé.

Zusammenfassung

Ziel dieser Studie ist es, zwei Methoden zu kommentieren, die angewendet werden, um die Prämien für Wiederauffüllungen von Schadenexzedenten-Rückversicherungverträgen zu schätzen. Für die pro rata capita Klausel analysieren wir die rate on line Methode. Für die pro rata temporis Klausel untersuchen wir die Anwendung einer einfachen Approximation. Der Effekt einer Jahresabzugsfranchise wird auch besprochen.

Kaspar Rufibach, Manuel Bertschy, Manuela Schüttel,
Michael Vock, Tina Wasserfallen, Bern

Eintrittsraten und Austrittswahrscheinlichkeiten EVK 2000

1 Einleitung, Ziel

Die zu erwartende Anzahl der Ein- und Austritte von Arbeitnehmern in der Vorsorgeeinrichtung eines Unternehmens (Turnover genannt) ist entscheidend für die Bewertung der Verpflichtungen. Die meisten internationalen Rechnungslegungsvorschriften berücksichtigen diesen Turnover. Für die Schweiz sind unseres Wissens bisher keine entsprechenden Zahlen verfügbar. Am Institut für mathematische Statistik und Versicherungslehre der Universität Bern beschäftigen sich deshalb die Teilnehmer des Seminars über Themen der Sozialversicherung, das unter der Leitung von Prof. Dr. Heinz Schmid stattfindet, mit der Erarbeitung solcher Zahlen. Der vorliegende Artikel gibt einen Einblick in erste Resultate. Die Seminararbeit (inkl. Tabellen in elektronischer Form) ist ab Mitte 2001 auf der Institutswebsite (www.imsv.unibe.ch) verfügbar.
Ziel war es, aufgrund der Angaben der Eidgenössischen Versicherungskasse (EVK) über die Jahre 1993 bis 1998 die erwartete Anzahl Ein- bzw. Austritte pro Jahr im Verhältnis zum aktuellen Risikobestand zu berechnen, und zwar getrennt nach Alter und Geschlecht. Diese Zahlen werden im folgenden als Eintrittsrate bzw. Austrittswahrscheinlichkeiten bezeichnet (bei den Eintritten handelt es sich nicht um Wahrscheinlichkeiten, weil die geschätzten Werte grösser als eins sein können). In der Regel werden nur Austritte betrachtet, die nicht auf versicherte Ereignisse zurückzuführen sind. Schätzungen von Sterbe- und Invalidierungswahrscheinlichkeiten sind hinlänglich bekannt, und der Altersrücktritt ist weitgehend deterministisch. Hierfür sei auf die Technischen Grundlagen EVK 2000 [1], die unter der Leitung von Prof. H. Schmid erarbeitet wurden, verwiesen. Wo für festgestellte Phänomene Erklärungen angegeben werden, sind diese nicht als eindeutig identifizierte Ursachen aufzufassen, sondern als mögliche Ausgangspunkte weiterer Untersuchungen.

2 Berechnungsgrundlagen

Die EVK hat den Autoren freundlicherweise die Grundzahlen der Jahre 1993 bis 1998 zur Berechnung der Technischen Grundlagen EVK 2000 [1] zur weiteren Auswertung überlassen. Im einzelnen sind dies:

- Risikobestände der Aktiven (für jedes beobachtete Jahr, nach Alter und Geschlecht)
- Anzahl der Eintritte (für jeden Monat, nach Alter und Geschlecht)
- Anzahl der Austritte, die nicht durch versicherte Ereignisse (Tod, Invalidität und Alter) begründet sind (für jedes Jahr, nach Alter und Geschlecht)

Durch den kollektiven Übertritt der Swisscom-Arbeitnehmer aus der EVK (im Rahmen der Privatisierung der Swisscom) in die neue, separate Swisscom-Pensionskasse (COMPLAN) wären die Austrittszahlen erheblich verfälscht worden (weil der Risikobestand über die Zeit nicht konstant gewesen wäre). Da diese Fälle aber in den vorhandenen Daten besonders gekennzeichnet waren, konnten sie von Anfang an eliminiert werden.
Im Sinne einer Plausibilitätskontrolle wurden die Daten verschiedener Jahre bzw. Monate anhand grafischer Darstellungen verglichen, was zu folgender Korrektur Anlass gab:
Generell weist der Januar überdurchschnittlich hohe Eintrittszahlen auf, was vielleicht auf besondere Anstellungsbedingungen beim Bund zurückzuführen ist. Hingegen konnten die Januar-Daten einiger Jahre nicht verwendet werden, da sie offensichtlich durch einmalige Ereignisse bzw. die Erhebungsmethode beeinflusst worden sind.

- Januar 1993: Für die Erstellung der Technischen Grundlagen EVK 2000 [1] wurden sämtliche bisherigen Versicherten am 01.01.93 als Eintrittsgeneration betrachtet und sind damit nicht von den echten Eintritten auf diesen Zeitpunkt unterscheidbar.
- Januar 1995: Die Zahlen dieses Monats liegen im gesamten Altersbereich bei Männern und Frauen massiv über dem Durchschnitt, so dass wir von einem besonderen Ereignis ausgehen müssen. Dieses Ereignis konnte leider noch nicht eindeutig ermittelt werden.
- Januar 1998: Hier waren die Eintrittsdaten nur bei den Frauen stark erhöht. Dabei ist die Integration eines ehemaligen Sonderbestandes (PTT-Sonderbestand C25) in den regulären Bestand der EVK eine denkbare Ursache.

Aufgrund dieser Beobachtungen wurde die Anzahl der Eintritte in den Monaten Januar 1993, 1995 und 1998 (bei den Frauen) bzw. Januar 1993 und 1995 (bei

den Männern) in erster Näherung mit dem Durchschnitt der übrigen Januar-Zahlen ersetzt.

3 Definitionen, Berechnungsmethoden

3.1 Berücksichtigter Altersbereich

Um verlässliche Zahlen präsentieren zu können, wurde der betrachtete Altersbereich folgendermassen eingeschränkt: Als untere Grenze wurde ein Alter von 25 gewählt. Dies aus der Überlegung heraus, dass nach BVG der Sparprozess für die Altersleistungen erst mit 25 beginnt. Die höchsten betrachteten Alter sind 57 (Frauen) resp. 60 (Männer), was in den untersuchten Jahren dem Alter fünf Jahre vor dem ordentlichen Rücktrittsalter entspricht. Höhere Alter zu betrachten erscheint nicht sinnvoll, da die Zahl der freiwilligen Austritte ohne Anspruch auf Versicherungsleistungen in diesem Altersbereich sehr klein ist.

3.2 Definition der verwendeten Masszahlen

Da eine standardisierte Symbolik für die betrachteten Grössen fehlt, definieren wir für unsere Zwecke geeignete Symbole. Diese beginnen mit T bzw. t für Turnover. Grossbuchstaben bezeichnen absolute Zahlen, Kleinbuchstaben bezeichnen Verhältnisse. I bzw. i steht für *Eintritte* (in), O bzw. o für *Austritte* (out), s für *Saldo*:

x Alter eines Mannes

x_s Schlussalter der Männer

TR_x Zahl der unter einjährigem Risiko gestandenen Männer im Alter x (Risikobestand)

TI_x Anzahl der während einem Jahr eintretenden Männer im Alter x

TO_x Anzahl der während einem Jahr austretenden Männer im Alter x, wobei Austritte wegen Tod, Invalidität und Alter nicht berücksichtigt werden

ti_x Anzahl der während einem Jahr eintretenden Männer im Alter x im Verhältnis zum Risikobestand (Eintrittsrate)

to_x Wahrscheinlichkeit eines x-jährigen Mannes, vor erreichen des Alters $x+1$ auszutreten, wobei Austritte wegen Tod, Invalidität und Alter nicht berücksichtigt werden (Austrittswahrscheinlichkeit)

ts_x Mutationssaldo der Männer im Alter x relativ zum Risikobestand

TL_x^{to} Ordnung der „Aktiven“, wobei Austritte wegen Tod, Invalidität und Alter nicht berücksichtigt werden

TL_x Ordnung der „Aktiven“, wobei Mutationen wegen Austritt, Tod und Invalidität berücksichtigt werden

tn_x^{to} mittlere Verbleibsdauer eines x-jährigen Mannes im Bestand, wobei Austritte wegen Tod, Invalidität und Alter nicht berücksichtigt werden

tn_x mittlere Aktivitätsdauer eines x-jährigen Mannes im Bestand, wobei Mutationen wegen Austritt, Tod und Invalidität berücksichtigt werden

${}^*q_x^a$ Wahrscheinlichkeit (abhängige) eines x-jährigen aktiven Mannes, das Alter $x+1$ als Aktiver nicht zu erleben [1]

${}^*i_x$ Wahrscheinlichkeit (abhängige) eines x-jährigen aktiven Mannes, innerhalb eines Jahres aus dem Bestand der Aktiven auszuscheiden und in den Bestand der Invalidenrentner überzutreten [1]

Für die Frauen gelten dieselben Bezeichnungen mit y statt x. Rohe (ungeglättete) Werte tragen auf dem entsprechenden Symbol zusätzlich ein „Dach“-Zeichen, also zum Beispiel $\hat{ti}_x$ statt ti_x.

3.3 Ausgleichsmethode und verwendete Software

Die rohen Werte $\hat{ti}_x$, $\hat{to}_x$ und $\hat{ts}_x$ wurden mit Hilfe von Splines ausgeglichen. Dazu wurde die Funktion „smooth.spline“ in S-PLUS verwendet. Die Inputparameter wurden so gewählt, dass eine vernünftige Glättung erzielt wurde, ohne jedoch einen allzugrossen Fehler einzuführen. Siehe dazu [3].

4 Ergebnisse

4.1 Eintrittsrate

Geht man von einem deterministischen Modell ohne Austritte aus, kann der Quotient

$$\frac{TI_x}{TR_x}$$

als Eintrittsintensität interpretiert werden. Die Entwicklung des Bestandes ist dann durch

$$TR_{x+dx} = TR_x e^{\frac{TI_x}{TR_x}dx}$$

gegeben, was uns zur Schätzung der Eintrittsrate mittels

$$\hat{ti}_x = e^{\frac{TI_x}{TR_x}} - 1$$

führt. Mit der Spline-Glättung ergibt sich daraus die ausgeglichene Eintrittsrate ti_x.
Bei beiden Geschlechtern zeigt sich, dass zwischen den Eintrittsraten verschiedener Monate wesentliche Unterschiede bestehen (Grafiken 1, 2, 3, 4):
Am meisten fallen die – trotz vorgenommener Korrektur nach wie vor hohen – Werte im Januar auf. Dies bedürfte weiterer Abklärungen in Zusammenarbeit mit den Personaldiensten des Bundes. Eine mögliche Erklärung wäre, dass eine neue Arbeitsstelle oft zu Beginn eines neuen Jahres angetreten wird. Denkbar ist auch, dass durch Besoldungsanpassungen per ersten Januar verschiedene Personenkategorien neu unter das Obligatorium der beruflichen Vorsorge fallen.
Betrachten wir die Eintrittsraten nach Kalenderjahren, so scheinen diese mit den Jahren zuzunehmen, vor allem in jungen Altern (Grafiken 5, 6, 7, 8): Bis rund zum Alter 40 entspricht die Reihenfolge der Kurven genau jener der Kalenderjahre. Dies kann damit zusammenhängen, dass die Bereitschaft und sogar die Notwendigkeit, in jungen Jahren öfters die Stelle zu wechseln, in der Zeit von 1993 bis 1998 zugenommen hat. Auffallend ist weiter der unterschiedliche Kurvenverlauf der beiden Geschlechter (Grafik 9): Bei den Frauen nimmt die Eintrittsrate fast linear ab, wogegen die Kurve der Männer eher an einen exponentiellen Abfall erinnert. Möglicherweise hängt dies mit der höheren Anzahl weiblicher Wiedereinsteigerinnen zusammen.

4.2 Austrittswahrscheinlichkeiten

Diese werden nach Formel (11.3.7) in [2] folgendermassen geschätzt:

$$\hat{to}_x = 1 - e^{-\frac{TO_x}{TR_x}}$$

Mit dem Spline-Ausgleich ergeben sich die Werte to_x. Auffallend ist, dass die Werte im Jahr 1998 mit Abstand am höchsten sind. Das mit den Kalenderjahren ansteigende Niveau der Austrittswahrscheinlichkeiten lässt eine Konjunkturabhängigkeit vermuten. Die hohen Austrittswahrscheinlichkeiten der Frauen zwischen 25 und 35 Jahren resultieren wohl aus der Tatsache, dass in diesem Alter viele aus dem Berufsleben aussteigen, um eine Familie zu gründen. Auch bei den Austrittswahrscheinlichkeiten der Männer sieht man Schwankungen zwischen den Jahren. Grosse Unterschiede beobachtet man vor allem im Alter von 25 bis 35. (Grafiken 10, 11, 12, 13)

Die Austrittswahrscheinlichkeiten der Frauen sind wesentlich höher als jene der Männer. Es sieht so aus, dass die Frauen ihren Arbeitsplatz häufiger wechseln als die Männer. In den höheren Altern könnte dies auch mit einer früheren freiwilligen Aufgabe der Erwerbstätigkeit ohne Rentenbezug zusammenhängen (Grafik 14).

4.3 Mittlere Verbleibsdauer

Die mittlere Verbleibsdauer wird folgendermassen geschätzt: Aus der Beziehung für die unabhängige (nach [5]) Ordnung der Aktiven

$$TL_{x+1}^{to} = TL_x^{to}(1 - to_x)$$

ergibt sich:

$$tn_x^{to} = \frac{1}{TL_x^{to}} \sum_{t=0}^{s_x - x} TL_{x+t}^{to} - \frac{1}{2}$$

Aus den Austrittswahrscheinlichkeiten lässt sich berechnen, wie lange eine x-jährige Person im Durchschnitt in der Vorsorgeeinrichtung verweilt ohne Berücksichtigung der Austritte infolge Tod, Invalidität und Alter. Aufgrund der Daten kann man sagen, dass die Frauen früher die Vorsorgeeinrichtung verlassen als die Männer. Dies hängt wohl mit der Familiengründung zusammen. Die Werte im Jahr 1998 liegen am tiefsten, was sich aus den in diesem Jahr höchsten Austrittswahrscheinlichkeiten ergibt (Grafiken 15, 16, 17).

4.4 Mittlere Aktivitätsdauer

Bei der Berechnung der mittleren Aktivitätsdauer sind drei Ausscheideursachen zu berücksichtigen: Tod, Invalidität und freiwilliger Austritt, das heisst, die mittlere Aktivitätsdauer ist eine abhängige Ordnung (siehe [5]). Die dazugehörigen „abhängigen" Wahrscheinlichkeiten berechnen sich wie folgt (nach [4]):

$$^{\circ}q_x^a := {}^{*}q_x \left(1 - \frac{1}{2} to_x\right)$$

$$^{\circ}i_x^a := {}^{*}i_x \left(1 - \frac{1}{2} to_x\right)$$

$$^{\circ}to_x^a := to_x \left(1 - \frac{1}{2}({}^{*}q_x + {}^{*}i_x)\right)$$

Damit folgt nach Formel 31 im ersten Teil von [5]:

$$TL_{x+1} = TL_x(1 - {}^\circ to_x - {}^\circ q_x^a - {}^\circ i_x)$$

und daraus berechnet sich schliesslich die mittlere Aktivitätsdauer:

$$tn_x = \frac{1}{TL_x} \sum_{t=0}^{s_x - x} TL_{x+t} - \frac{1}{2}$$

Die mittlere Aktivitätsdauer ist im jüngeren Alter bei Frauen und Männern kürzer als die mittlere Verbleibsdauer. Im höheren Alter besteht kein sichtbarer Unterschied zwischen den Verlaufskurven (Grafik 18).

5 Vergleich Eintrittsraten und Austrittswahrscheinlichkeiten, relativer Mutationssaldo

5.1 Vergleich Ein- und Austritte

Beim Gesamtüberblick Eintritte/Austritte nach Geschlechtern (Grafik 19) stechen hauptsächlich zwei Besonderheiten ins Auge: Erstens sind die Eintrittsraten und die Austrittswahrscheinlichkeiten der Frauen ab 30 für alle Alter höher als jene der Männer, bei den Frauen ist also die Fluktuation stärker. Zweitens sind die einzelnen Werte für die Männer ab etwa 50 praktisch Null. Für die Frauen sind jedoch die Werte bei unserem gewählten Schlussalter (57) noch deutlich positiv. Dies könnte mit einer vermehrten Frühpensionierung von Männern – im Gegensatz zu einem freiwilligen Rückzug der Frauen aus dem Erwerbsleben – zusammenhängen.

5.2 Mutationssaldo

Zum Schluss versuchen wir noch, Erkenntnisse über die Nettozu- oder -abnahme des Bundespersonals zu erhalten (Grafik 20). Dazu betrachten wir die Grösse

$$\hat{ts}_x = \frac{TI_x - TO_x}{TR_x}$$

also das Verhältnis der Nettobestandesänderung (Eintritte – Austritte) zum Risikobestand. Auch diese rohen Werte glätten wir wieder mit Splines. Als erstes erkennt man, dass im Mittel über die sechs betrachteten Jahre die betrachtete Grösse – ausser bei den Frauen über 54 – positiv ist. Das heisst, dass mehr

Personen ein- als ausgetreten sind. Dies ist offensichtlich, da die Austritte durch Invalidität und Tod durch Neueintritte kompensiert werden müssen. Weiter fällt auf, dass – gemessen am Risikobestand – die Nettobestandeszunahme bei den jungen Männern stärker ist als bei den jungen Frauen. Das Bild ändert sich ab Alter 35, wo dann über einen Zeitraum von ca. 15 Jahren die Zunahme bei den Frauen grösser ist als bei den Männern. Dies könnte wiederum auf eine hohe Zahl Wiedereinsteigerinnen hinweisen, die jene der Frauen, die den Bundesdienst verlassen, übersteigt.

6 Schlussbemerkungen

Die Schätzungen der Eintrittsraten und Austrittswahrscheinlichkeiten sowie der daraus abgeleiteten Grössen basieren auf verhältnismässig umfangreichem Zahlenmaterial. Es zeigt sich aber, dass sich die Werte im betrachteten Zeitraum von sechs Jahren recht erheblich verändern. Auch Besonderheiten der EVK (bzw. des Bundes als Arbeitgeber) dürften die Zahlen stark beeinflussen. Für die praktische Verwendung der Schätzungen kommt deshalb der Anpassung an die Verhältnisse einer bestimmten Branche und eines bestimmten Unternehmens, an die Wirtschaftslage usw. noch eine wesentlich grössere Bedeutung zu, als dies etwa bei der Wahl von Sterbetafeln der Fall ist.

Der Dank der Autoren gilt Frau M.-T. Kohler für ihre fachliche Unterstützung.

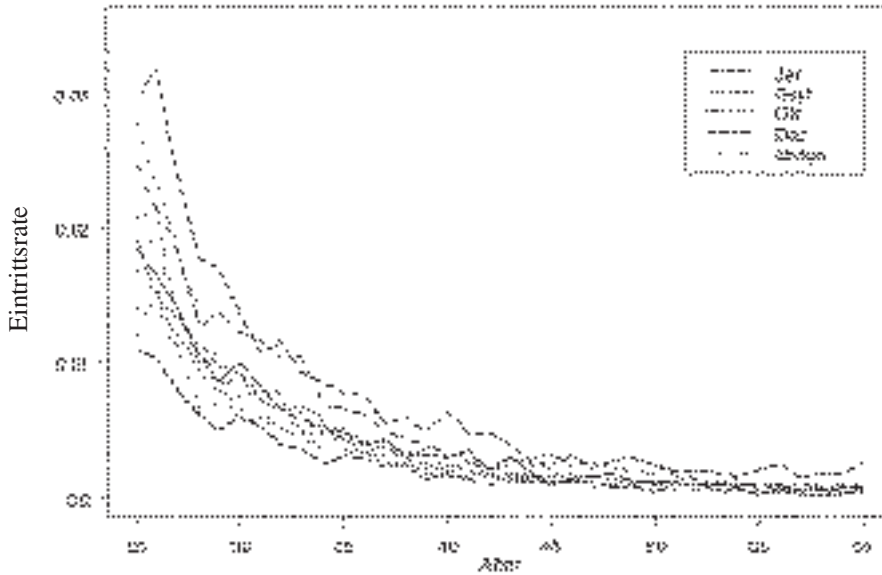
Tabelle A: Eintrittsraten und Austrittswahrscheinlichkeiten Männer EVK 2000

x	ti_x	to_x	$^\circ q_x$	$^\circ i_x$	$^\circ to_x$	TL_x^{to}	TL_x	tn_x^{to}	tn_x
25	0.2601	0.0717	0.0006	0.0005	0.0716	1.0000	1.0000	17.1852	16.5098
26	0.2248	0.0716	0.0006	0.0005	0.0715	0.9283	0.9273	17.4733	16.7650
27	0.1904	0.0712	0.0006	0.0005	0.0712	0.8619	0.8600	17.7814	17.0381
28	0.1590	0.0706	0.0006	0.0006	0.0706	0.8005	0.7978	18.1067	17.3266
29	0.1327	0.0695	0.0005	0.0006	0.0694	0.7440	0.7406	18.4444	17.6261
30	0.1118	0.0673	0.0005	0.0007	0.0672	0.6923	0.6883	18.7843	17.9274
31	0.0954	0.0637	0.0006	0.0008	0.0636	0.6457	0.6412	19.1032	18.2089
32	0.0826	0.0587	0.0006	0.0009	0.0586	0.6046	0.5995	19.3684	18.4397
33	0.0724	0.0527	0.0006	0.0010	0.0527	0.5691	0.5635	19.5442	18.5865
34	0.0639	0.0467	0.0007	0.0011	0.0466	0.5391	0.5329	19.6039	18.6243
35	0.0567	0.0411	0.0007	0.0012	0.0411	0.5140	0.5071	19.5389	18.5453
36	0.0504	0.0363	0.0008	0.0013	0.0362	0.4929	0.4854	19.3547	18.3551
37	0.0448	0.0322	0.0009	0.0013	0.0322	0.4750	0.4668	19.0641	18.0657
38	0.0398	0.0288	0.0009	0.0014	0.0288	0.4597	0.4507	18.6817	17.6911
39	0.0354	0.0260	0.0010	0.0015	0.0260	0.4464	0.4367	18.2215	17.2443
40	0.0317	0.0237	0.0011	0.0016	0.0236	0.4348	0.4242	17.6949	16.7361
41	0.0285	0.0216	0.0012	0.0017	0.0215	0.4245	0.4131	17.1116	16.1753
42	0.0258	0.0196	0.0013	0.0018	0.0196	0.4154	0.4030	16.4780	15.5679
43	0.0236	0.0178	0.0013	0.0019	0.0178	0.4072	0.3938	15.7981	14.9175
44	0.0218	0.0162	0.0014	0.0021	0.0161	0.4000	0.3855	15.0756	14.2278
45	0.0203	0.0147	0.0015	0.0023	0.0146	0.3935	0.3780	14.3149	13.5027
46	0.0190	0.0134	0.0016	0.0026	0.0133	0.3877	0.3710	13.5205	12.7469
47	0.0177	0.0122	0.0017	0.0029	0.0122	0.3825	0.3645	12.6969	11.9648
48	0.0166	0.0113	0.0018	0.0033	0.0112	0.3779	0.3584	11.8480	11.1604
49	0.0156	0.0104	0.0019	0.0039	0.0103	0.3736	0.3525	10.9772	10.3373
50	0.0146	0.0096	0.0021	0.0045	0.0095	0.3697	0.3469	10.0869	9.4985
51	0.0137	0.0088	0.0022	0.0054	0.0088	0.3662	0.3413	9.1795	8.6462
52	0.0128	0.0081	0.0025	0.0065	0.0081	0.3630	0.3356	8.2567	7.7826
53	0.0118	0.0075	0.0027	0.0079	0.0075	0.3600	0.3299	7.3203	6.9091
54	0.0109	0.0070	0.0030	0.0095	0.0069	0.3573	0.3239	6.3719	6.0270
55	0.0101	0.0064	0.0034	0.0115	0.0064	0.3548	0.3176	5.4130	5.1367
56	0.0094	0.0060	0.0038	0.0140	0.0059	0.3526	0.3109	4.4448	4.2377
57	0.0088	0.0056	0.0043	0.0168	0.0055	0.3504	0.3035	3.4686	3.3285
58	0.0085	0.0051	0.0049	0.0203	0.0051	0.3485	0.2954	2.4852	2.4060
59	0.0084	0.0046	0.0057	0.0243	0.0046	0.3467	0.2865	1.4954	1.4655
60	0.0084	0.0041	0.0063	0.0288	0.0040	0.3451	0.2766	0.5000	0.5000

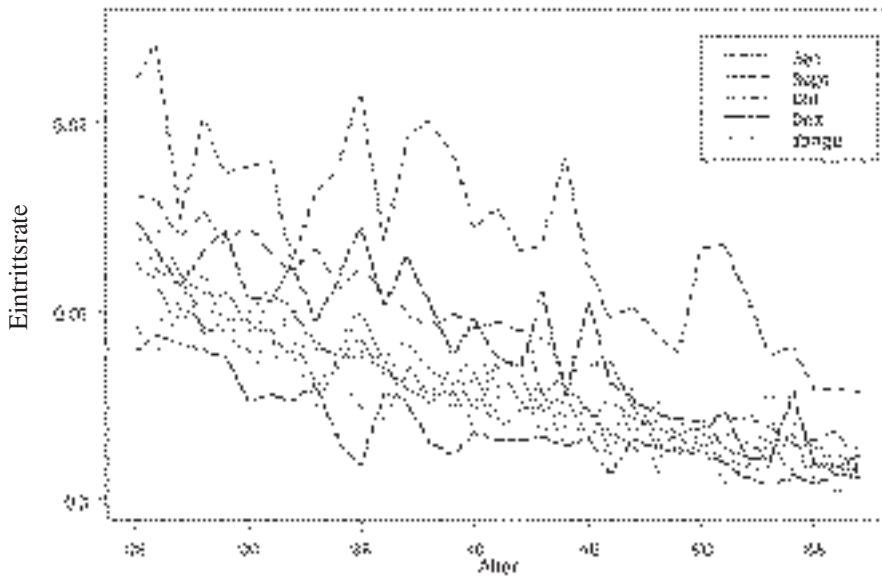
Tabelle B: Eintrittsraten und Austrittswahrscheinlichkeiten Frauen EVK 2000

y	ti_y	to_y	$^\circ q_y$	$^\circ i_y$	$^\circ to_y$	TL_y^{to}	TL_y	tn_y^{to}	tn_y
25	0.1428	0.0964	0.0003	0.0007	0.0963	1.0000	1.0000	10.6062	10.2905
26	0.1369	0.1016	0.0003	0.0009	0.1015	0.9036	0.9026	10.6841	10.3469
27	0.1310	0.1063	0.0004	0.0010	0.1062	0.8118	0.8099	10.8353	10.4741
28	0.1253	0.1100	0.0004	0.0011	0.1099	0.7255	0.7228	11.0649	10.6767
29	0.1199	0.1116	0.0004	0.0013	0.1115	0.6457	0.6422	11.3711	10.9530
30	0.1148	0.1104	0.0004	0.0014	0.1103	0.5737	0.5696	11.7364	11.2860
31	0.1102	0.1067	0.0003	0.0015	0.1066	0.5103	0.5057	12.1313	11.6475
32	0.1061	0.1006	0.0003	0.0017	0.1005	0.4559	0.4509	12.5204	12.0037
33	0.1027	0.0929	0.0003	0.0018	0.0928	0.4100	0.4046	12.8655	12.3182
34	0.0999	0.0845	0.0003	0.0020	0.0844	0.3719	0.3662	13.1316	12.5580
35	0.0973	0.0762	0.0003	0.0022	0.0761	0.3405	0.3345	13.2969	12.7024
36	0.0942	0.0687	0.0003	0.0024	0.0686	0.3145	0.3082	13.3529	12.7439
37	0.0908	0.0622	0.0004	0.0026	0.0621	0.2929	0.2862	13.3015	12.6845
38	0.0872	0.0566	0.0004	0.0028	0.0565	0.2747	0.2676	13.1501	12.5316
39	0.0837	0.0518	0.0004	0.0030	0.0517	0.2592	0.2516	12.9085	12.2946
40	0.0805	0.0478	0.0005	0.0033	0.0477	0.2458	0.2378	12.5859	11.9825
41	0.0775	0.0447	0.0005	0.0035	0.0446	0.2340	0.2255	12.1930	11.6057
42	0.0746	0.0422	0.0006	0.0038	0.0421	0.2235	0.2145	11.7404	11.1743
43	0.0713	0.0402	0.0007	0.0042	0.0401	0.2141	0.2046	11.2356	10.6954
44	0.0674	0.0384	0.0008	0.0045	0.0383	0.2055	0.1954	10.6847	10.1746
45	0.0628	0.0369	0.0008	0.0049	0.0368	0.1976	0.1869	10.0915	9.6155
46	0.0580	0.0358	0.0009	0.0054	0.0357	0.1903	0.1789	9.4591	9.0207
47	0.0536	0.0351	0.0010	0.0058	0.0350	0.1835	0.1714	8.7917	8.3941
48	0.0498	0.0348	0.0012	0.0064	0.0347	0.1771	0.1642	8.0937	7.7393
49	0.0469	0.0347	0.0013	0.0070	0.0345	0.1709	0.1573	7.3676	7.0584
50	0.0447	0.0345	0.0014	0.0077	0.0343	0.1650	0.1506	6.6142	6.3514
51	0.0426	0.0341	0.0015	0.0085	0.0340	0.1593	0.1440	5.8325	5.6167
52	0.0403	0.0338	0.0017	0.0094	0.0336	0.1538	0.1377	5.0209	4.8518
53	0.0378	0.0334	0.0018	0.0104	0.0332	0.1487	0.1316	4.1789	4.0549
54	0.0349	0.0332	0.0020	0.0116	0.0329	0.1437	0.1256	3.3061	3.2240
55	0.0314	0.0330	0.0021	0.0131	0.0328	0.1389	0.1197	2.4023	2.3569
56	0.0277	0.0328	0.0023	0.0147	0.0325	0.1343	0.1140	1.4672	1.4504
57	0.0240	0.0324	0.0026	0.0167	0.0321	0.1299	0.1083	0.5000	0.5000

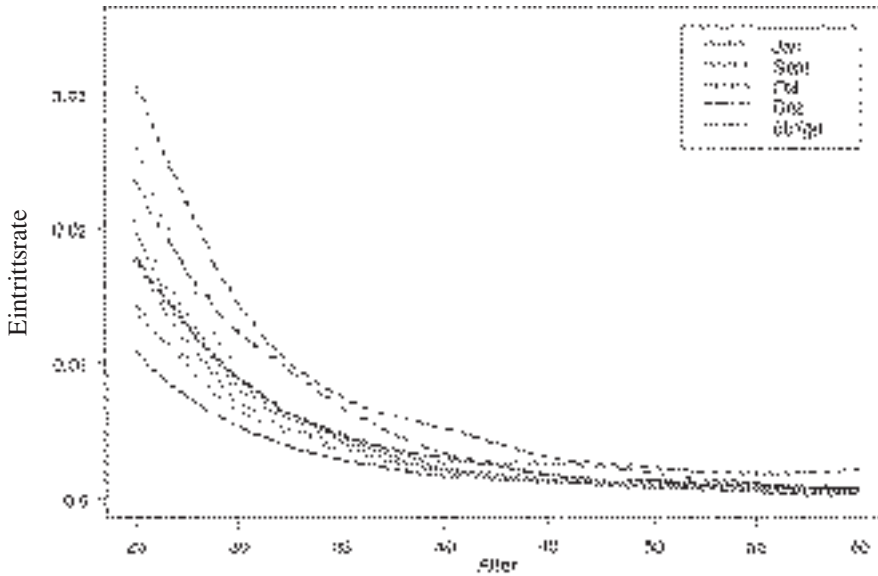
Grafik 1: Eintrittsraten Männer: Rohe Werte nach Monaten



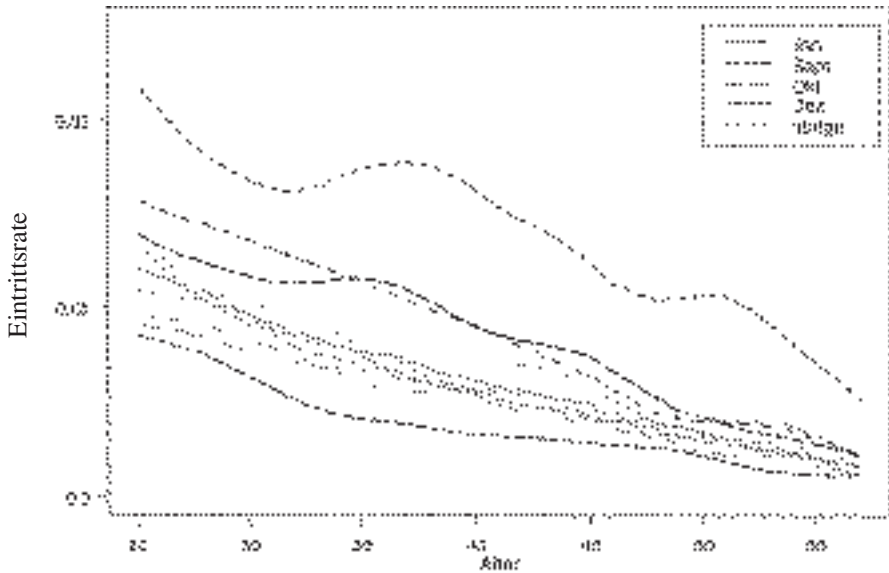
Grafik 2: Eintrittsraten Frauen: Rohe Werte nach Monaten



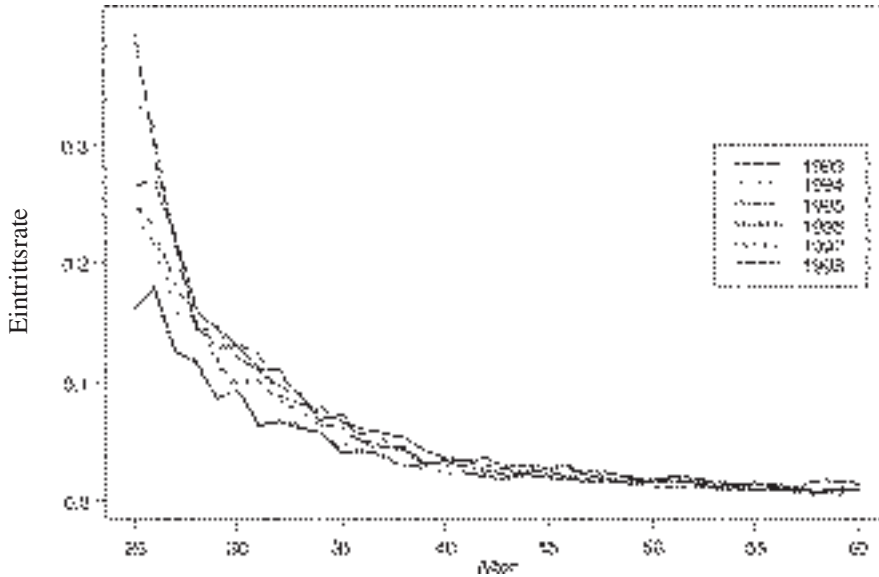
Grafik 3: Eintrittsraten Männer: Ausgleich nach Monaten



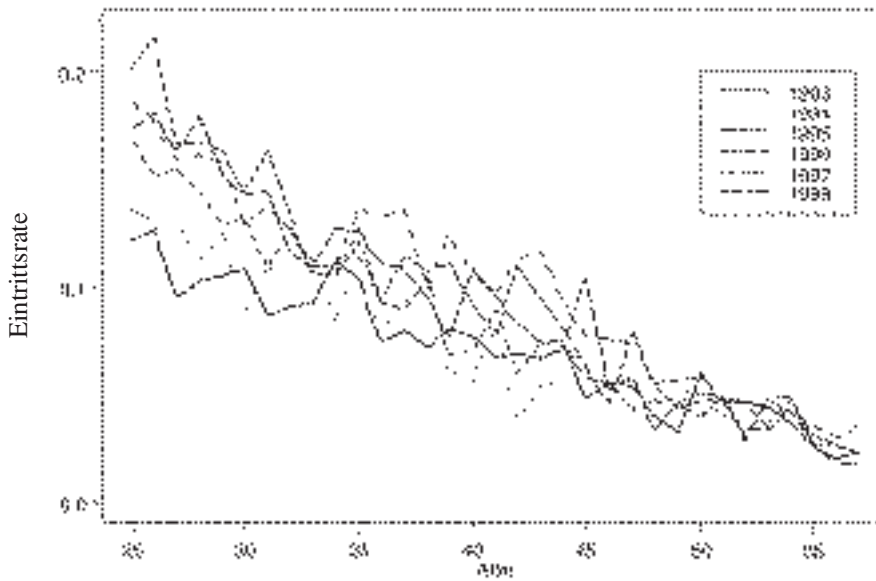
Grafik 4: Eintrittsraten Frauen: Ausgleich nach Monaten



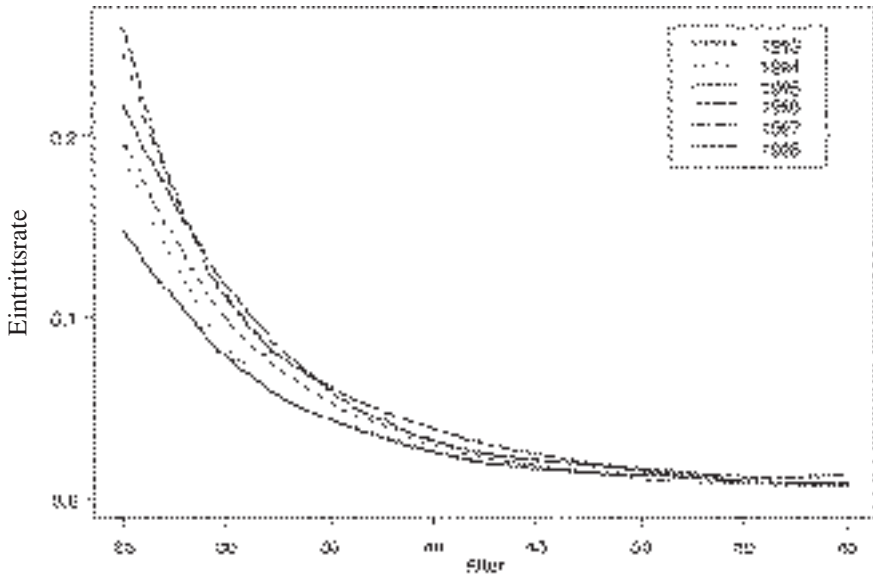
Grafik 5: Eintrittsraten Männer: Rohe Werte nach Jahren



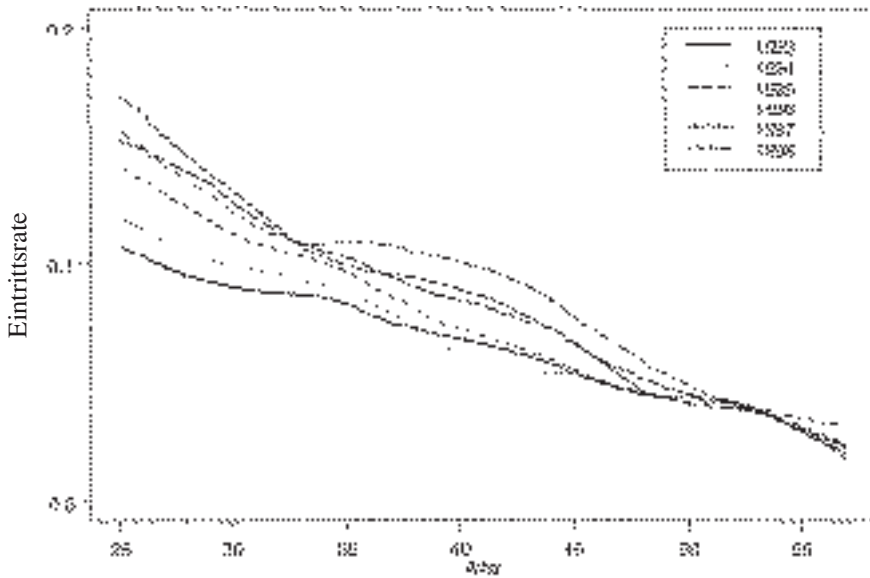
Grafik 6: Eintrittsraten Frauen: Rohe Werte nach Jahren



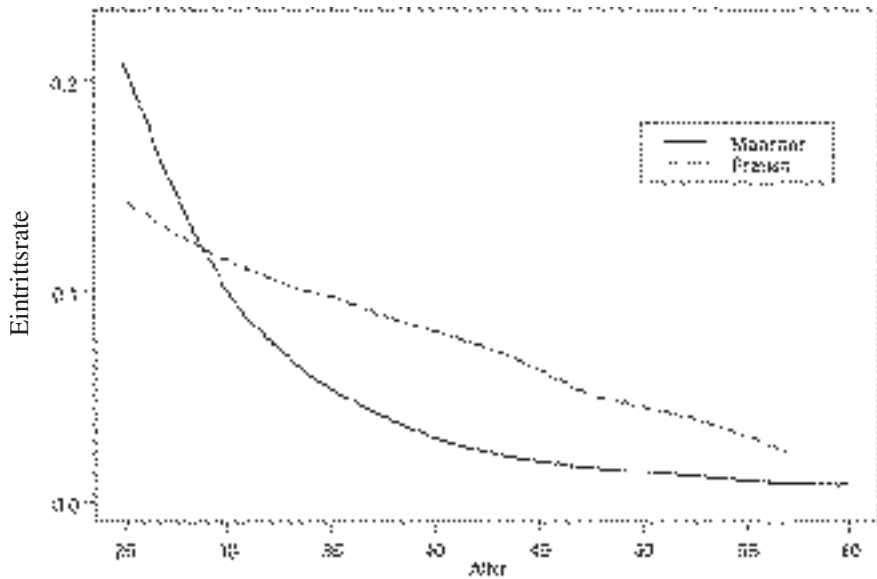
Grafik 7: Eintrittsraten Männer: Ausgleich nach Jahren



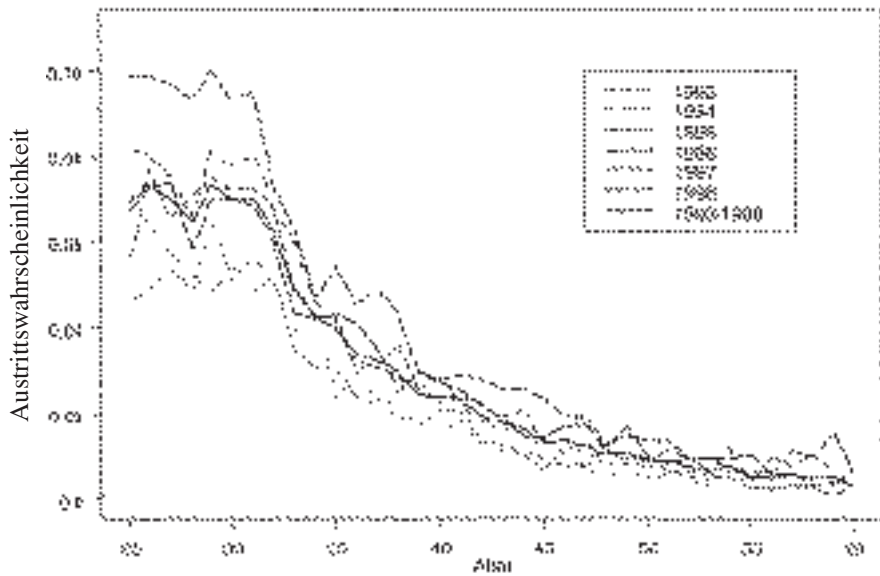
Grafik 8: Eintrittsraten Frauen: Ausgleich nach Jahren



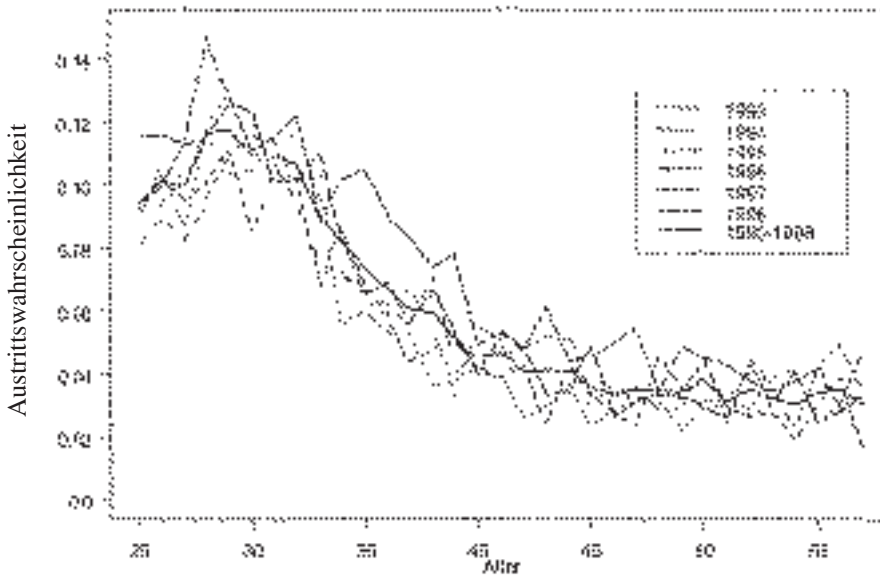
Grafik 9: Eintrittsraten: Vergleich Männer-Frauen Ausgleich



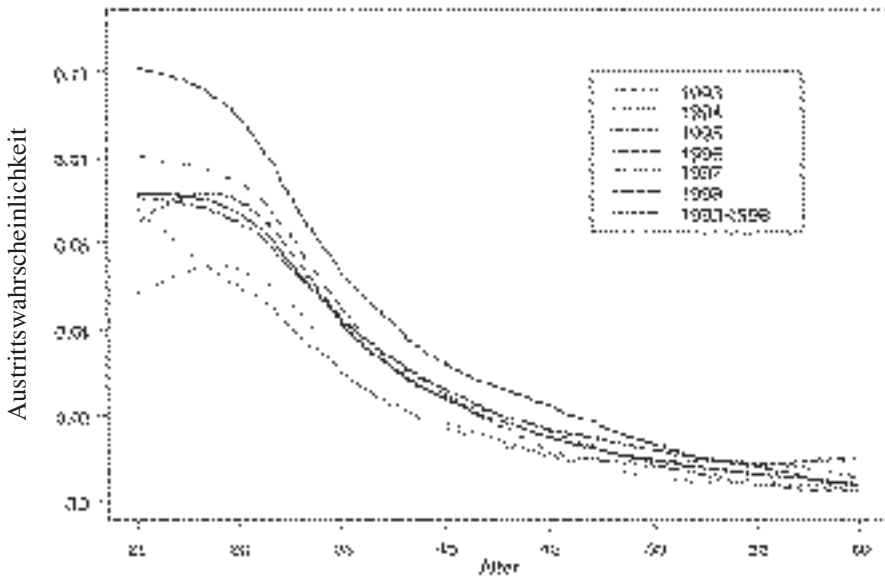
Grafik 10: Austrittswahrscheinlichkeiten Männer: Rohe Werte nach Jahren



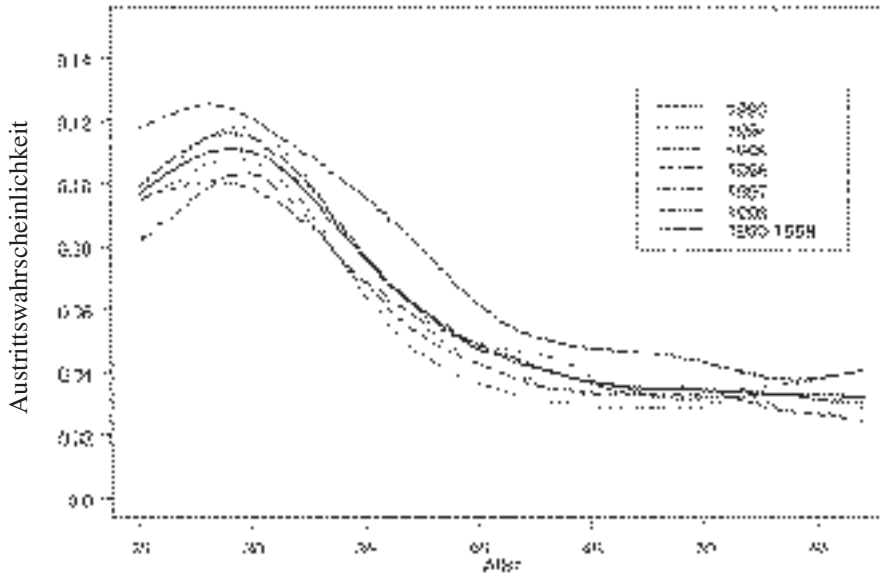
Grafik 11: Austrittswahrscheinlichkeiten Frauen: Rohe Werte nach Jahren



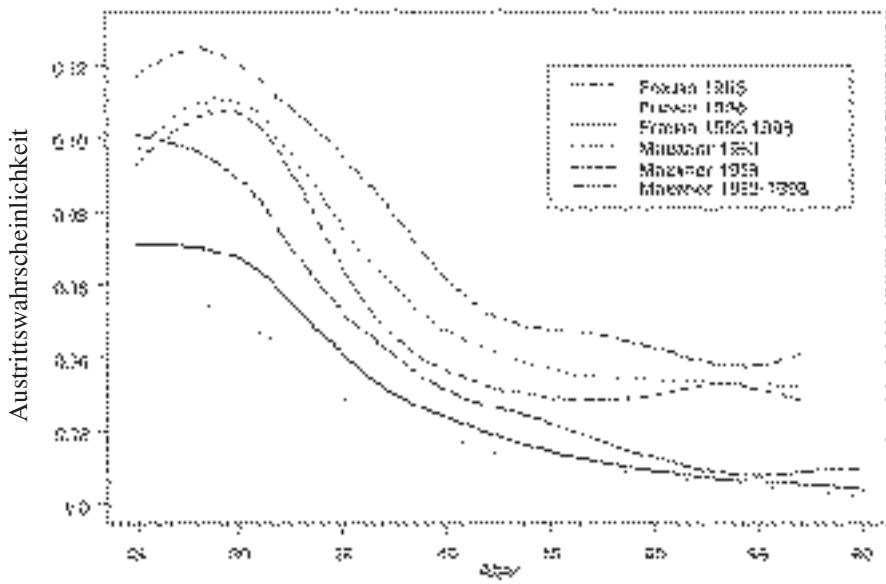
Grafik 12: Austrittswahrscheinlichkeiten Männer: Ausgleich nach Jahren



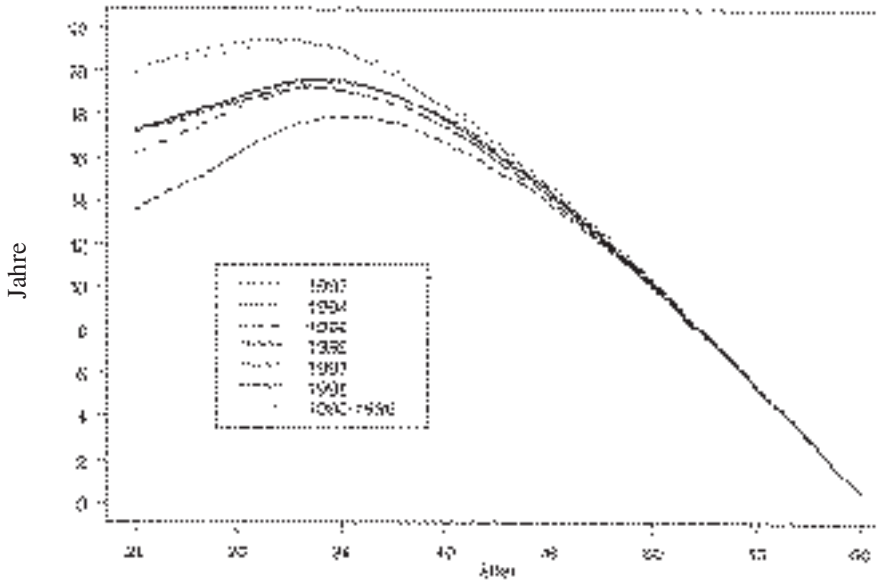
Grafik 13: Austrittswahrscheinlichkeiten Frauen: Ausgleich nach Jahren



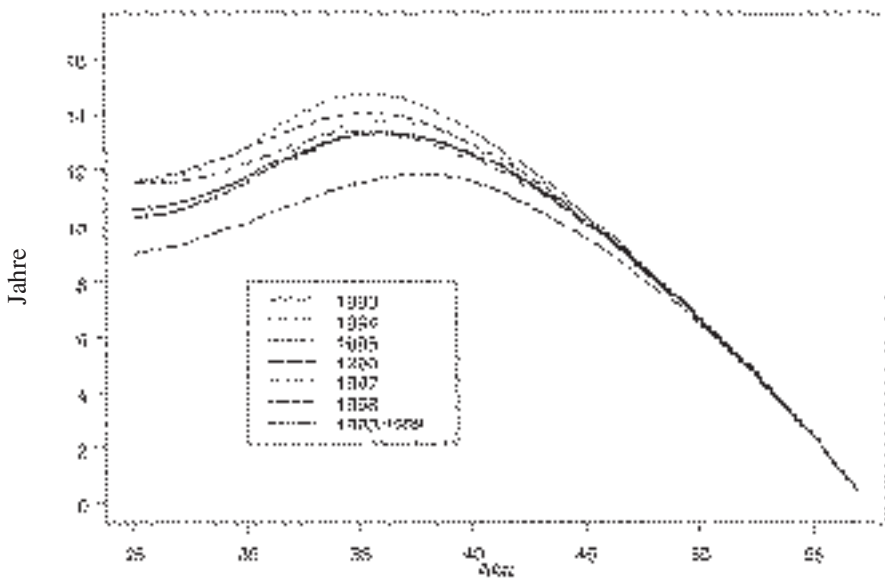
Grafik 14: Vergleich Austrittswahrscheinlichkeiten Frauen und Männer



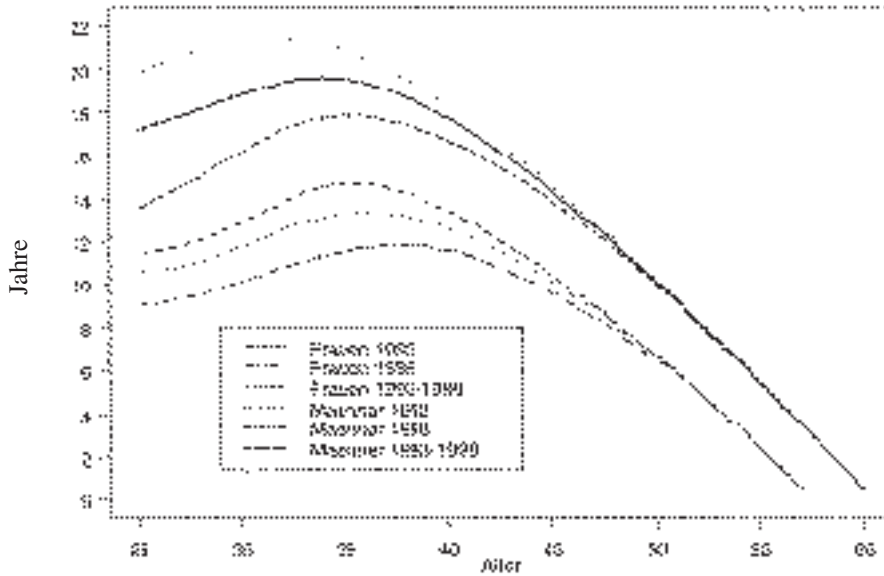
Grafik 15: Verbleibsdauer Männer nach Jahren



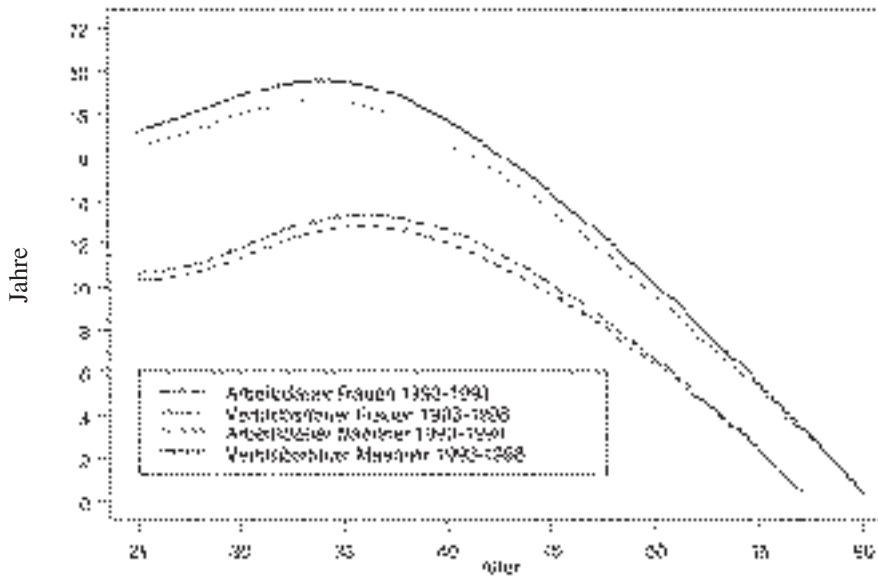
Grafik 16: Verbleibsdauer Frauen nach Jahren



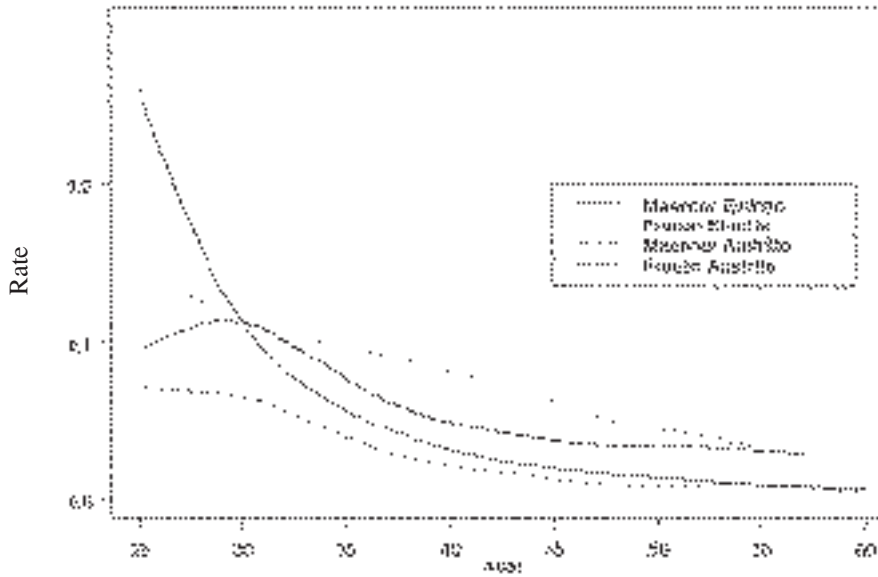
Grafik 17: Vergleich Verbleibsdauer Frauen und Männer



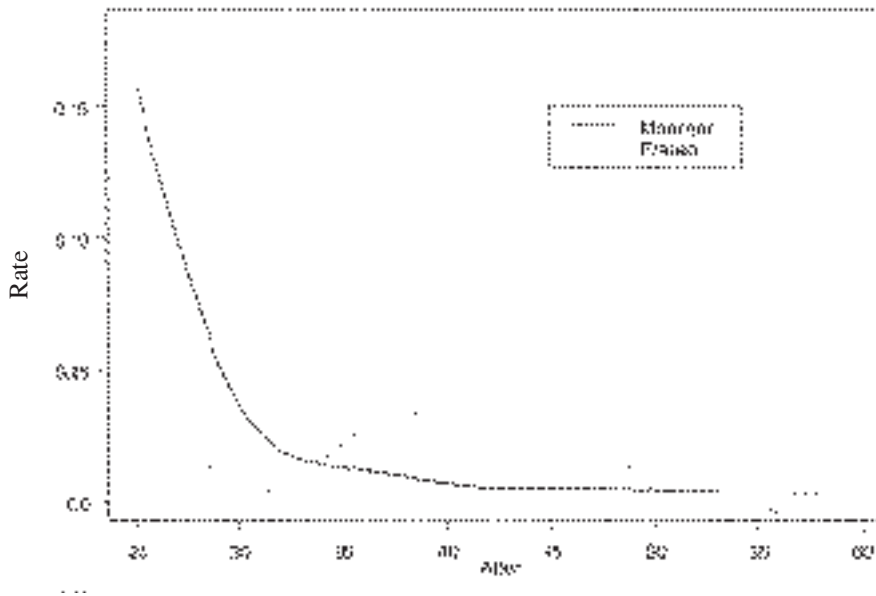
Grafik 18: mittlere Verbleibsdauer und mittlere Aktivitätsdauer Frauen und Männer



Grafik 19: Vergleich Männer-Frauen: Eintritte-Austritte



Grafik 20: Vergleich Mutationssaldo Frauen und Männer



Literatur

[1] Eidgenössische Versicherungskasse (2000) *Technische Grundlagen der eidgenössischen Versicherungskasse EVK 2000*. Eidgenössische Versicherungskasse / PUBLICA, Bern

[2] Gerber, H. U. (1995) *Life Insurance Mathematics*. Springer Verlag, Berlin, Heidelberg, New York

[3] Vogel, Franz (1983) Die Sterblichkeit bei den Einzelkapitalversicherungen der Schweizerischen Lebensversicherungs- und Rentenanstalt in den Jahren 1971 bis 1980, *Mitteilungen der Schweiz. Vereinigung der Versicherungsmathematiker*, 29–51

[4] Wagner, Christian (2000) Berechnungen für das IAS 19 im Rahmen der EVK 90, *Mitteilungen der Schweiz. Aktuarvereinigung*, 41–46

[5] Zwinggi, Ernst (1945) *Versicherungsmathematik*. Birkhäuser AG, Basel

Kaspar Rufibach
Institut für mathematische Statistik und Versicherungslehre
Sidlerstrasse 5
CH-3012 Bern

Zusammenfassung

Der Artikel stellt auf der Basis der Rohdaten zu den kürzlich erschienenen Technischen Grundlagen der Eidgenössischen Versicherungskasse (EVK 2000) die Berechnung und den Ausgleich der erwarteten Anzahl Ein- und Austritte in einer Pensionskasse sowie verwandter Grössen vor. Solche Angaben werden mit der Einführung internationaler Rechnungslegungsnormen zunehmend gebraucht.

Résumé

Utilisant les données brutes des bases techniques de la caisse fédérale d'assurance (EVK 2000) qui viennent de paraître, l'article présente le calcul et l'ajustement du nombre attendu d'entrées et de sorties dans une caisse de pension, ainsi que des grandeurs leur étant apparantées. L'importance de telles données s'accroit avec l'introduction de normes comptables internationales.

Summary

Based on the raw data of the recently appeared technical basics of the Federal Insurance Fund (EVK 2000), the paper describes the calculation and smoothing of the expected number of entrances and leavings in a pension plan and of related numbers. Such data is increasingly needed with the introduction of international accounting standards.