Market Data and Market Risk Modeling

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Introduction

As actuaries, we should have a critical look at models that are in use to quantify the risks to which insurers are exposed.

Market risk is one of the dominant drivers acting on the balance sheet.

We propose a family of random variables apt to deal with heavy tails and skewness in the financial data.

Recap on FINMA's SST standard model of market risks

Valuation model with the following basic modules

- Equities (and the likes) with underlying risk factors stock return, FX return
- Future Cash Flows with underlying risk factors continuous interest rates, spreads, FX return

Additionally, a module for equity- and FX-forward contracts and a delta term for all other financial instruments.

Recap of standard model

Notation

Set of currencies $\mathcal{F} = \{\text{CHF, EUR, USD, GBP, JPY}, \ldots\}$.
Set of equity-like instruments denominated in currency f,
$\mathcal{E}_f = \{\text{Stock, Real Estate, Commodity}, \ldots\}$ represented by Indices
Set of rating classes for bonds (and bond-like instruments),
$\mathcal{R}_f = \{\text{Govi, AAA}, ..., \text{BBB}\}$ (specific selection of ratings might depend on f)
Stochastic one-year change of continuous risk free interest rate in currency f for tenor t, ϱ_{ft}
Stochastic one-year change of logarithmic conversion rate from currency f to CHF, ζ_f (for $f = \text{CHF}$, we have $\zeta_f = 0$ a.s.)
Stochastic one-year log return of equity-like index i denominated in currency f, r_{fi}
Stochastic one-year change of spread of rating class p in currency f, s_{fp} (assumed to be constant for all tenors).

Recap of standard model

Given initial market values of the equities and the likes $S_{fi}(0)$ and future cash flows $C_{fp}(t)$ of tenors $t \in \{1, 2, \ldots\}$, both converted to CHF, the stochastic value of the portfolio (assets and liabilities) at the end of the year reads:

$$PL = \sum_{f \in \mathcal{F}, i \in \mathcal{E}_f} K_{fi} \cdot S_{fi}(0) \cdot e^{r_{fi} + \zeta_f} + \sum_{t \geq 1} \sum_{f \in \mathcal{F}, p \in \mathcal{R}_f} K_{fpt} \cdot C_{fp}(t) \cdot e^{-(\varrho_{ft} + s_{fp})t + \zeta_f}$$

where the terms

$$K_{fi} = 1/E[e^{r_{fi} + \zeta_f}],\ K_{fpt} = 1/E[e^{-(\varrho_{ft} + s_{fp})t + \zeta_f}].$$

guarantee a kind of stationarity.

- Omitted are the equity and currency forwards and the Delta term.

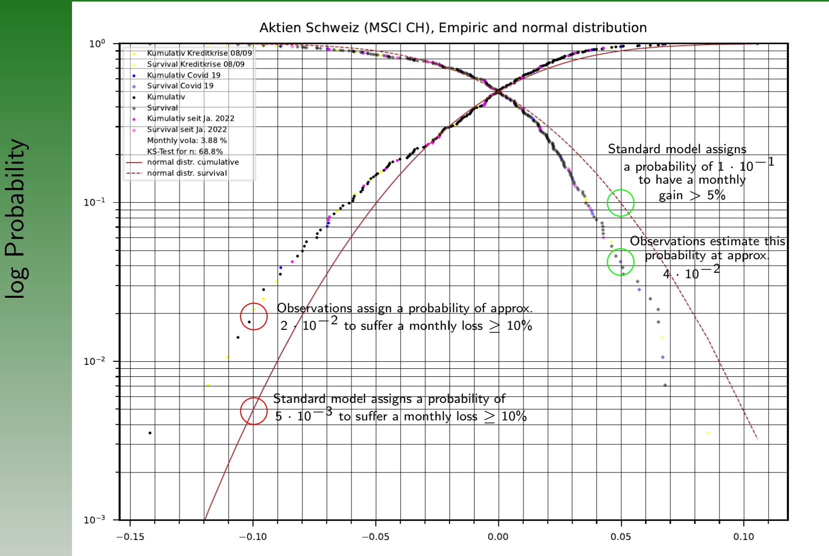
Normality assumption

Standard Model of FINMA: Centered normal distributions of risk factors.

Parameter estimation by standard moment estimators (unbiased vola and correlation for monthly returns of RF).

Compare data to modeled returns: Tail plots

Monthly Data: Rising a headache to the quant. risk manager?



Data source: Bloomberg

Obtain a better fit

The t-distribution (univariate case)

$X = \frac{Z}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} Y_i^2}}$, where $Z, Y_1, \ldots, Y_n \sim \mathcal{N}(0,1)$ (i.i.d)

More generally, for $\nu \in \mathbb{R}_{\geq 1}$:

$$X = \frac{Z}{\sqrt{1/\nu \cdot W}},$$

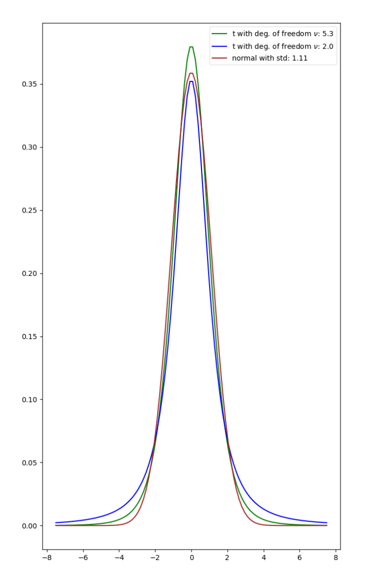
where $W \sim \chi^2_\nu$ and Z, W independent.
Density of X:

$$f_\nu(x) = C_\nu \cdot \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \; x \in \mathbb{R},$$

with the normalizing constant

$$C_\nu = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu} \cdot \Gamma\left(\frac{\nu}{2}\right)}.$$

Comments on the t-distribution



Family that interpolates between the arctan law ($\nu = 1$) and the normal law ($\nu \to \infty$)

L^1 if $\nu > 1$, L^2 if $\nu > 2$.

Tails are dangerous, Pareto-style

Symmetric densities however, market data typically are skewed.

For market risk modeling purposes t-distributed variables X have to be scaled: For $s > 0$ the density of sX is

$$f(x) = \frac{1}{s} f_\nu(\frac{x}{s}).$$

Obtain a better fit to market data

The branched t-distribution

Profit-Loss-data from financial markets are typically skewed (see the tail plots later).

Introduce skewness:

Given scalings $s_1, s_2 > 0$, degrees of freedom $\nu_1, \nu_2 > 1$, we can treat the lower and the upper branch of the t-distribution separately to obtain an asymmetric variable:

$$X = \frac{s_1 \cdot Z}{\sqrt{1/\nu_1 \cdot W_1}} \cdot I_{Z<0} + \frac{s_2 \cdot Z}{\sqrt{1/\nu_2 \cdot W_2}} \cdot I_{Z \geq 0},$$

where $Z \sim \mathcal{N}(0,1)$, $W_j \sim \chi^2_{\nu_j}$, and W_j, Z independent ($j = 1, 2$).

By tinkering near zero, it can be made absolutely continuous.

The branched t-distribution: Multivariate Version

Vector of risk factors $(X_1, \ldots, X_n)$ is given by components

$$X_i = \frac{s_{1i} \cdot Z_i}{\sqrt{1/\nu_{1i} \cdot W_{1i}}} \cdot I_{Z_i < 0} + \frac{s_{2i} \cdot Z_i}{\sqrt{1/\nu_{2i} \cdot W_{2i}}} \cdot I_{Z_i \geq 0},$$

where

$Z := (Z_1, \ldots, Z_n) \sim \mathcal{N}(0, \Sigma)$, with pos. def. correlation matrix Σ.

$W_{ji} \sim \chi^2_{\nu_{ji}}$ comonotonic for $i = 1, \ldots, n, j = 1, 2$ and

$(W_{11}, \ldots, W_{1n}, W_{21}, \ldots, W_{2n})$ and Z independent.

The parameters s_{ji}, ν_{ji} and Σ have to be estimated from the data.

Branched t: Parameter Estimation

Transform Monthly Indices

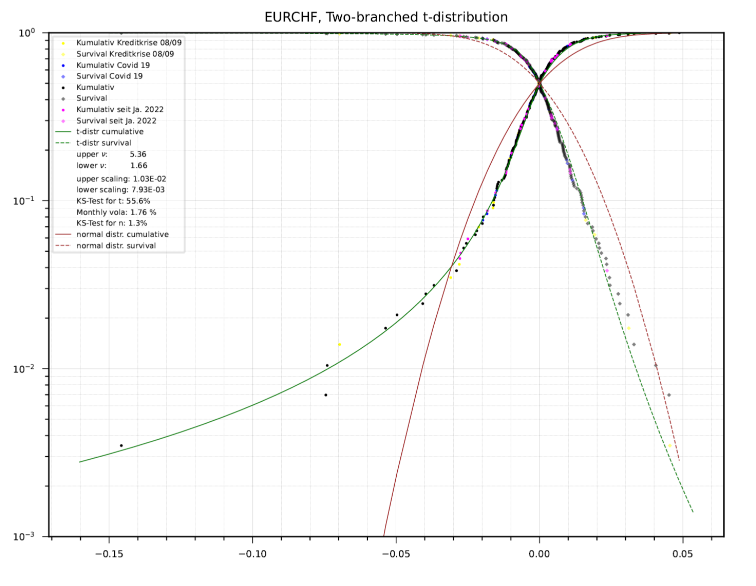
- Log returns for equities and the likes
- Additive increments of continuous interest rates
- Additive increments of spreads

Shift data so that the median is zero

MML and LS estimation of s_{ji}, ν_{ji}: Lower and upper branch can be treated separately.

Currently, Σ is estimated by Spearman's ρ and normalized.

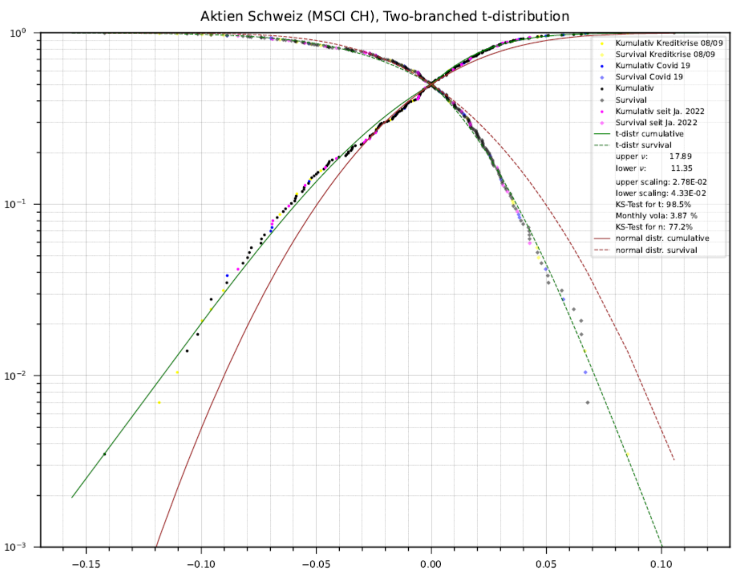
Example from monthly data



LS-estimators

Data source: Bloomberg

Example from monthly data



LS-estimators

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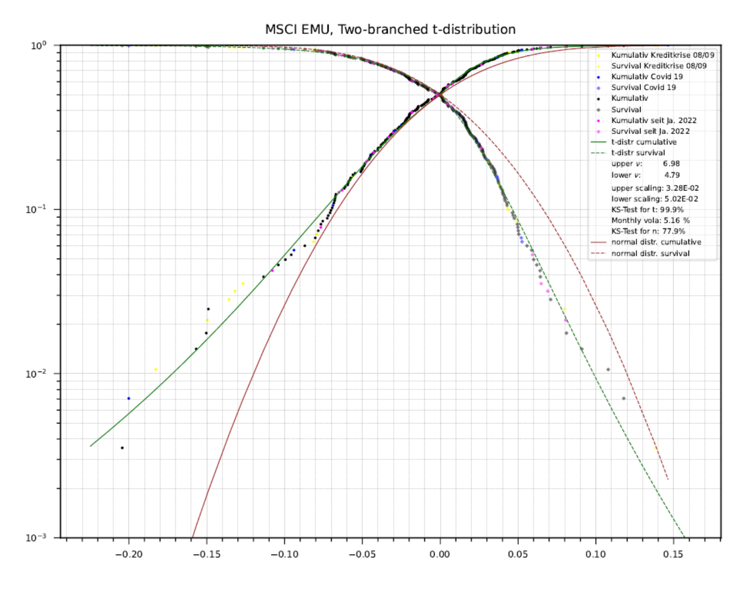
Example from monthly data



LS-estimators

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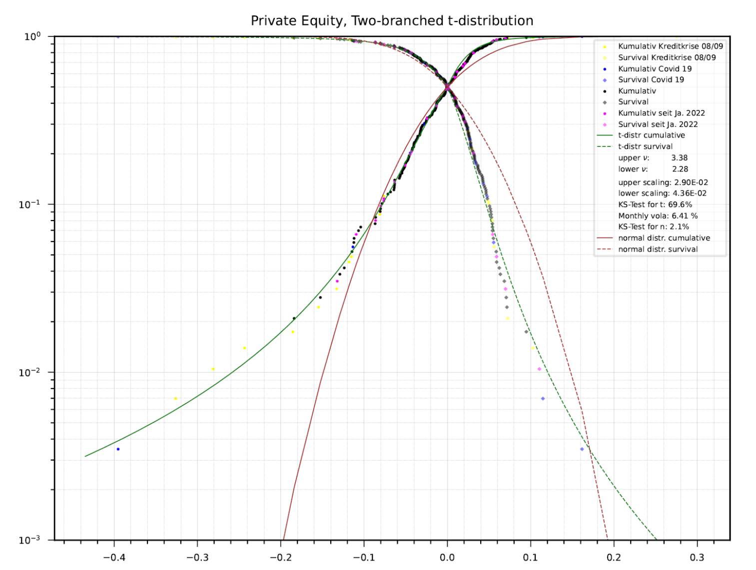
Example from monthly data



LS-estimators

Data source: Bloomberg

Passage from monthly to yearly data

Brief answer to headache question: It depends.

For a one-year return need to sum up 12 monthly returns under IID-assumption.

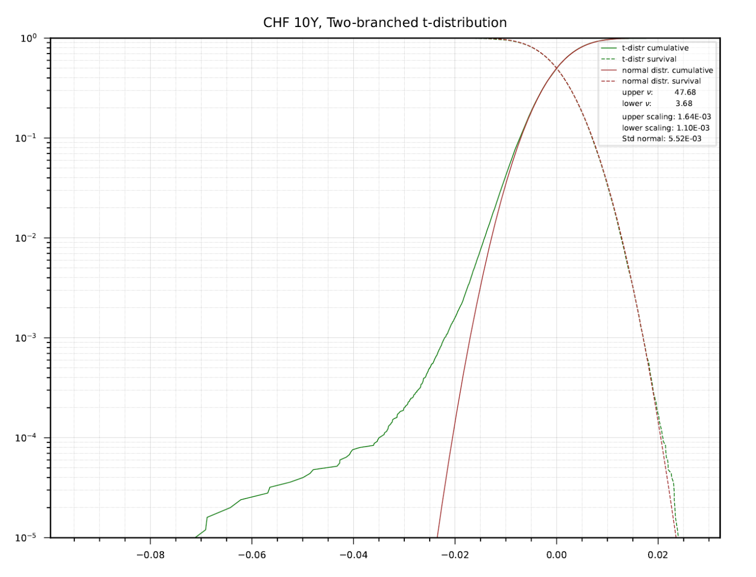
Branched t-variables do not form an infintely divisable family: Any help from the Berry-Esseen Theorem?

Note that if X is branched t with parameters $s_{1/2}, \nu_{1/2}$, then

$$\begin{aligned} E[X] &= s_2 \tfrac{\nu_2}{\nu_2-1} C_{\nu_2} - s_1 \tfrac{\nu_1}{\nu_1-1} C_{\nu_1}, \text{ if } \nu_{1/2} > 1 \\ E[X^2] &= \tfrac{1}{2}[(s_1)^2 \tfrac{\nu_1}{\nu_1-2} + (s_2)^2 \tfrac{\nu_2}{\nu_2-2}], \text{ if } \nu_{1/2} > 2 \\ E[X^3] &= 2 s_2^3 \tfrac{\nu_2^2}{(\nu_2-1)(\nu_2-3)} C_{\nu_2} - 2 s_1^3 \tfrac{\nu_1^2}{(\nu_1-1)(\nu_1-3)} C_{\nu_1}, \text{ if } \nu_{1/2} > 3. \end{aligned}$$

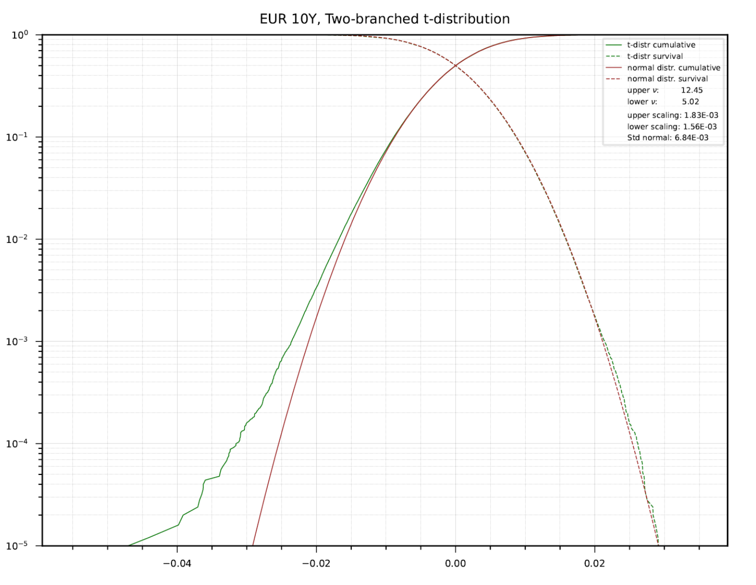
If the ν's are high enough, convergence to normal law is fairly good. However, if they are close to 2 - or less - convergence can be bad. See the next slides.

Yearly data: LS-estimators



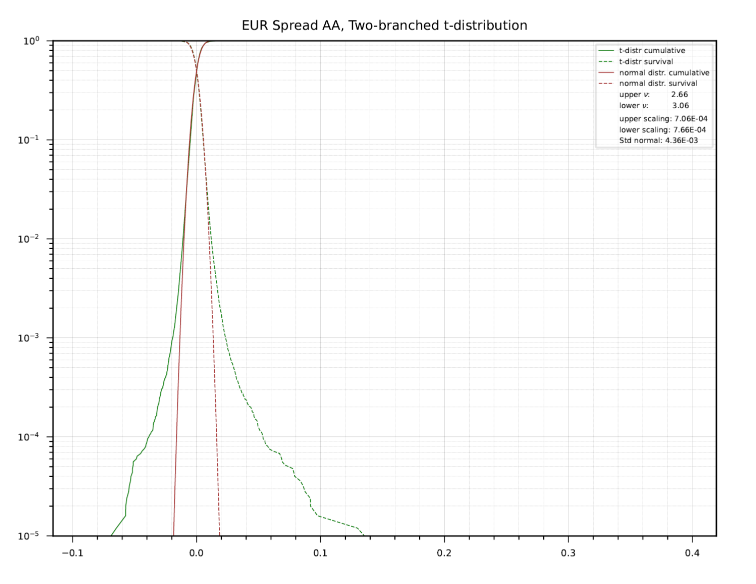
Data source: SNB

Yearly data: LS-estimators



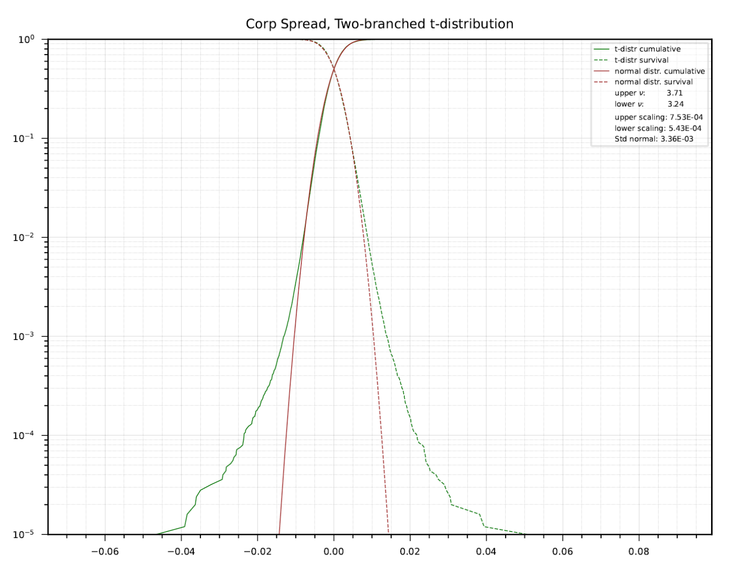
Data source: Bloomberg

Yearly data: LS-estimators



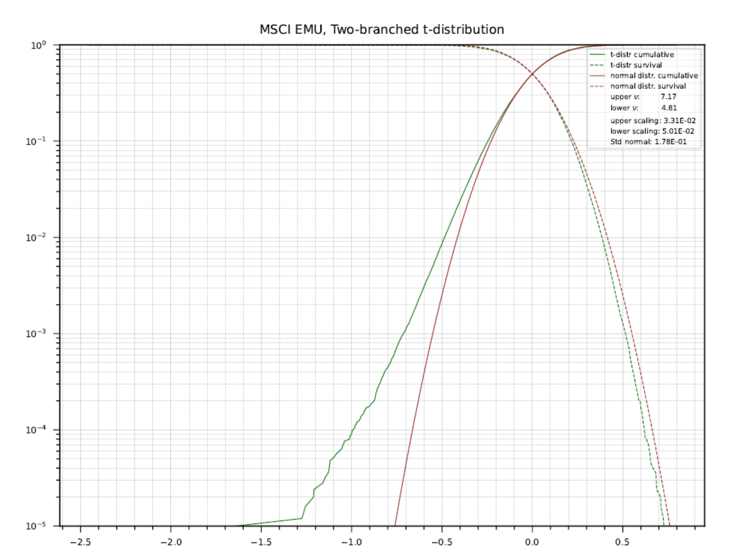
Data source: Bloomberg

Yearly data: LS-estimators



Data source: SNB

Yearly data: LS-estimators



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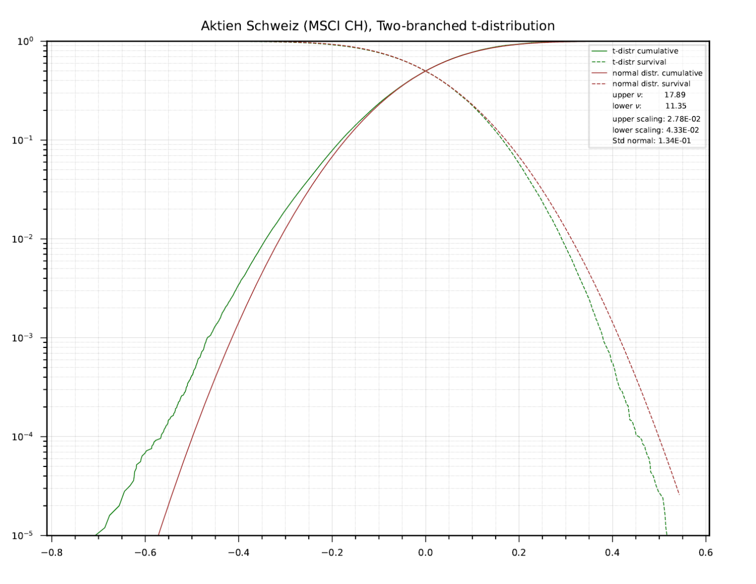
Yearly data: LS-estimators



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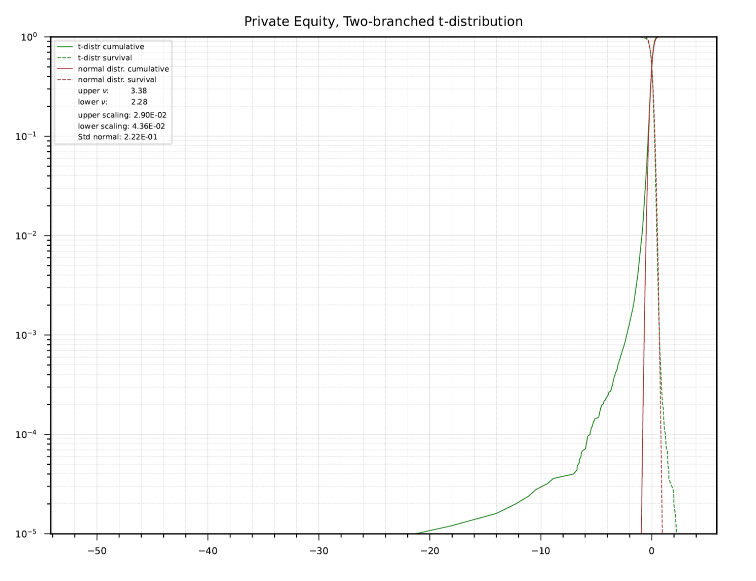

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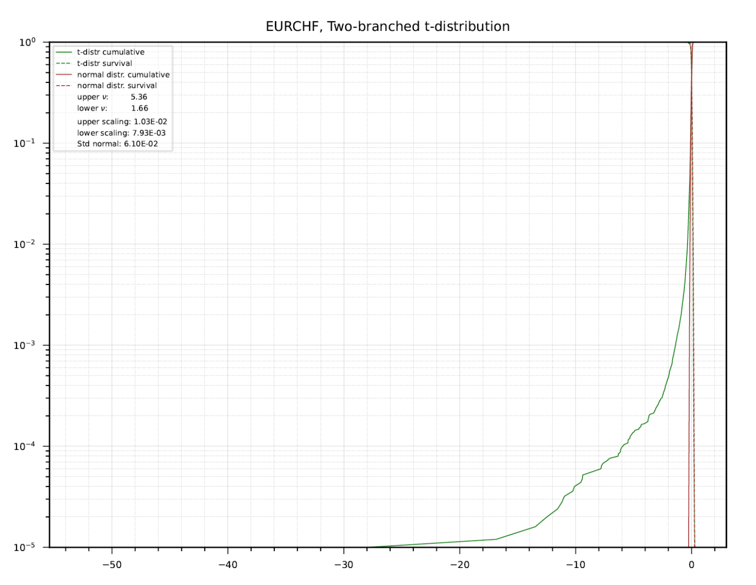
Yearly data: LS-estimators



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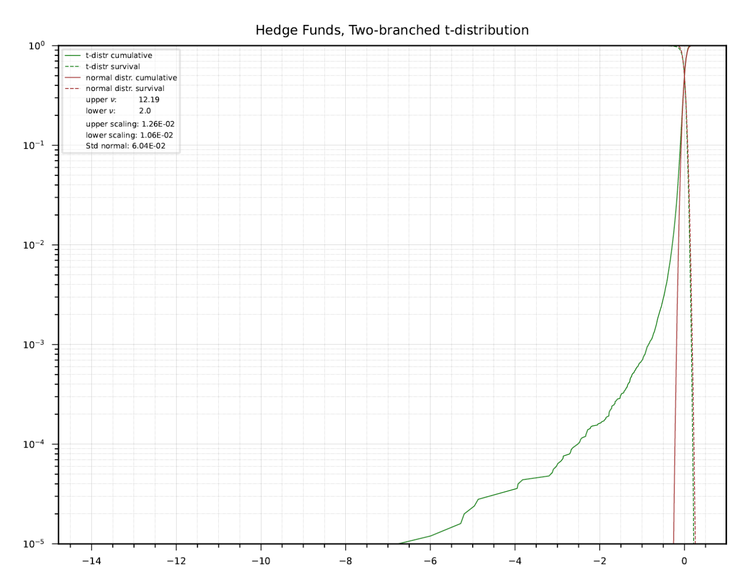
Yearly data: LS-estimators



Data source: Bloomberg

Branched t distributions: Final comments

For some of the risk factors normality assumption OK: EUR IR 10Y, FX JPY-CHF, Gold (?)

For some it is bad: Equity, PE, HF, ReaL Estate, and spreads.

Degrees of freedom can be very sensitive w.r.t. small variations of monthly returns.

Estimated parameters from a bootstrap simulation can show considerable variation.

Bayesian Modeling for parameters to be estimated, apriori assumptions suggest $s_j \sim \mathcal{N}(s_{j0}, \sigma_j^2), \nu_j \sim \Gamma(\alpha_j, \beta_j, l_j)$

Use MCMC to get the posterior distributions.

Illustrative Example

Asset type	currency	Exposure in Mio. CHF
participation	CHF	-
equity	CHF	293.2
equity	EUR	38.0
equity	USD	244.9
equity	GBP	17.1
equity	JPY	67.6
hedge fund	CHF	55.0
hedge fund	EUR	0.1
hedge fund	USD	1.1
hedge fund	GBP	0.1
hedge fund	JPY	0.7
private equity	CHF	15.6
private equity	EUR	46.6
private equity	USD	79.6
private equity	GBP	0.0
private equity	JPY	37.5
real estate private	CHF	576.9
real estate commercia	CHF	419.7
others	USD	95.6
others	USD	74.6
GOVI	CHF	922.7
CANT	CHF	1'741.6
CORP	CHF	969.3
AAA	CHF	4.6
AA	CHF	185.4
A	CHF	145.7
BBB	CHF	89.9
A	EUR	2.5
BBB	EUR	124.1
Total assets		6'249.6
Total Insu. Liabilities		5'112.0

Purely fictional company

No delta term

No forwards

Illustrative Example, Results

Market risk of a fictional company

	stand alone risk		
	t-distr (250k Sim)	n-distr (250k Sim)	rel. Diff t:n)
All Risk Factors	907.3	730.1	24%
Zinsen CHF	629.5	553.8	14%
Zinsen EUR	11.6	11.1	4%
Spreads	188.1	170.0	11%
FX Risk	200.2	154.9	29%
Aktien	219.5	201.7	9%
Immo	139.6	123.9	13%
Hedge Fund Risk	13.1	14.7	-11%
Private Equity Risk	128.7	86.3	49%
Andere	38.1	39.6	-4%

HF-Risk: In SST, estimated vola is doubled.

FX-Risk is important on a stand-alone basis, however it diversifies well.

Conclusions

Look at the data Compare empirical distributions to modeled distributions to understand the differences.

Time intervals Understand the passage from monthly to yearly events.

Detect optimistic modeling Normal distributions tend to underestimate tail events also on a yearly basis.

Try various estimators MML has good asymptotic properties; LS is apt to accurately estimate tails.

Inform investment committees on risk factors that are particularly heavy tailed (e.g. Private Equity), or difficult to model.

Think about dependency modeling

Appendix: MML-Estimators

Let $x_1 < x_2 < \ldots x_{N_1} < 0 < x_{N_1+1} < \ldots < x_{N_2}$: Available observations (monthly increments)
For $j = 1, 2$, we denote by f_j the density of the s_j-scaled t-distribution with ν_j degrees of freedom, that is,

$$f_j(\nu_j, s_j; x) = C_{\nu_j} \cdot \frac{1}{s_j}\left(1 + \frac{1}{\nu_j}\frac{x^2}{s_j^2}\right)^{-\frac{\nu_j+1}{2}}, \; x \in \mathbb{R}.$$

Then the **Maximum-Likelihood estimator** (MML) of $(\nu_j, s_j), j = 1, 2$, is given by the solution of the optimization problems

$j = 1$, lower branch: $\operatorname{argmin}_{\nu_1, s_1} - \sum_{i=1}^{N_1} \ln f_1(\nu_1, s_1; x_i)$,

$j = 2$, upper branch: $\operatorname{argmin}_{\nu_2, s_2} - \sum_{i=N_1+1}^{N_2} \ln f_2(\nu_2, s_2; x_i)$.

Appendix: LS-Estimators

Let $q(\nu_j, \ldots) : (0,1) \to \mathbb{R}$ be the quantile function of the t-distribution with ν_j degrees of freedom.

Then the **Least-square estimator** (LS) of $(\nu_j, s_j), j = 1, 2$, is given by the (unique) solution of the optimization problems

$$j = 1, \quad \text{lower branch:} \quad \text{argmin}_{\nu_1, s_1} \sum_{i=1}^{N_1} [s_1 \cdot q(\nu_1, i/N_2) - x_i]^2,$$

$$j = 2, \quad \text{upper branch:} \quad \text{argmin}_{\nu_2, s_2} \sum_{i=N_1+1}^{N_2} [s_2 \cdot q(\nu_2, i/N_2) - x_i]^2.$$

Least square optimization taking place on the x-axis (rather than on the y-axis).