Risk Bounds under Uncertainty and Model Risk

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Risk Measures
A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is a function mapping random variables to real numbers.

**Applications in finance and insurance:**

- regulatory capital requirement
- capital allocation
- insurance pricing
- ...
For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

Value-at-Risk:

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$
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**Range-Value-at-Risk:**

$$\text{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_u(X) \, du.$$
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**Value-at-Risk:**

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

**Range-Value-at-Risk:**

$$\text{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_\alpha^\beta \text{VaR}_u(X) \, du.$$

**Expected Shortfall:**

$$\text{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(X) \, du.$$
VaR, RVaR, ES

Risk Bounds under Uncertainty and Model Risk

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Properties for risk assessment:
[Artzner et al., 1999, Föllmer & Schied, 2011]
law-invariant, monotone, convex, sub-additive, coherent, translation invariant, ...

Statistical properties:
[Gneiting, 2011, Krätschmer et al., 2014, Pesenti et al., 2016]
elicitable, backtestable, robust, ...

Risk assessment under uncertainty:
bounds for risk measures, worst-case risk measures, aggregation robustness, rearrangement algorithm, joint mixability, ...
Distributional uncertainty
Risk assessment in the presence of uncertainty:

- distributional uncertainty
- parameter uncertainty
- distributional misspecifications
- data collection

What are the possible values of

\[ \rho(X), \quad \text{if } X \in \mathcal{M}, \]

for an uncertainty set \( \mathcal{M} \).
**Distributional uncertainty**

*Best-case and worst-case risk measures*

$$\rho(X) = \inf_{X \in M} \rho(X), \quad \overline{\rho(X)} = \sup_{X \in M} \rho(X).$$

**Risk measure bounds:**

$$\rho(X) \in (\underline{\rho(X)}, \overline{\rho(X)})$$
VaR and ES

\[ \text{VaR}_\alpha \quad \hat{\text{VaR}}_\alpha \quad \overline{\text{VaR}}_\alpha \]

\[ \text{ES}_\alpha \quad \hat{\text{ES}}_\alpha \quad \overline{\text{ES}}_\alpha \]
An uncertainty set $\mathcal{M}$ describes the knowledge about the uncertainty in the distribution of $X$. 
An uncertainty set $\mathcal{M}$ describes the knowledge about the uncertainty in the distribution of $X$.

For example: “(nearly) complete uncertainty”

$$\mathcal{M}(\mu, \sigma) = \left\{ X \mid E(X) = \mu, \text{Var}(X) = \sigma^2 \right\}$$
Bounds with moment constraints

\[ \text{VaR}_\alpha(X) \text{ bounds} \]
\[ \left[ \mu - \sigma \sqrt{\frac{1 - \alpha}{\alpha}}, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right] \]

\[ \text{RVaR}_{\alpha,\beta}(X) \text{ bounds} \]
\[ \left[ \mu - \sigma \sqrt{\frac{1 - \beta}{\beta}}, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right] \]

\[ \text{ES}_\alpha(X) \text{ bounds} \]
\[ \left[ \mu, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right] \]
Bounds with moment constraints

VaR$_\alpha$(X) bounds

$$\left[ \mu - \sigma \sqrt{\frac{1 - \alpha}{\alpha}}, \quad \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

RVaR$_{\alpha,\beta}$(X) bounds

$$\left[ \mu - \sigma \sqrt{\frac{1 - \beta}{\beta}}, \quad \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

ES$_\alpha$(X) bounds

$$\left[ \mu, \quad \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

extremely large
Bounds with moment constraints

$\text{VaR}_\alpha(X)$ bounds

$$\left[ \mu - \sigma \sqrt{\frac{1 - \alpha}{\alpha}}, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

$\text{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[ \mu - \sigma \sqrt{\frac{1 - \beta}{\beta}}, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

$\text{ES}_\alpha(X)$ bounds

$$\left[ \mu, \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]$$

! extremely large ! independent of the distribution of $X$
Bounds with moment constraints

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\left[ \mu, \quad \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \right]
\]

! extremely large ! independent of the distribution of \( X \)

! worst-case distribution is a two point distribution.
Bounds with moment constraints

<table>
<thead>
<tr>
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<tr>
<td>VaR_{0.975}</td>
<td>9.68</td>
<td>13.92</td>
<td>14.46</td>
</tr>
<tr>
<td>RVaR_{0.95,0.99}</td>
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<td>13.82</td>
<td>14.33</td>
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<td>10.00</td>
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\( X \) has mean 10 and standard deviation 2.

\[
\begin{align*}
\text{VaR}_{0.975} & = 9.68 & 13.92 & 14.46 & 22.49 \\
\text{RVaR}_{0.95,0.99} & = 9.80 & 13.82 & 14.33 & 18.72 \\
\text{ES}_{0.95} & = 10.00 & 14.13 & 14.79 & 18.72 \\
\end{align*}
\]
### Bounds with moment constraints

$X$ has mean $10$ and standard deviation $2$.

$\Rightarrow$ For any random variable, with mean $= 10$ and sd $= 2$, its VaR at level $0.975$ belongs to $(9.68, 22.49)$.

<table>
<thead>
<tr>
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Risk Bounds under Uncertainty and Model Risk

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VaR bounds; with mean 10 and sd 2
% of VaR bounds; with mean 10 and sd 2
ES bounds; with mean 10 and sd 2
% of ES bounds; with mean 10 and sd 2
Towards better bounds:

⇒ Include further knowledge to the uncertainty set $\mathcal{M}$. 
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- higher moments [Cornilly et al., 2018]
- symmetric distributions [Zhu & Shao, 2018, Li et al., 2018]
- unimodal distributions [Li et al., 2018].
Towards better bounds:

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⇒ only marginal improvements
⇒ worst-case distribution is a two point distribution
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► Wasserstein ball [Pesenti et al., 2020]
Let $X_0 \sim F_0$ be a reference distribution with mean $\mu$ and standard deviation $\sigma > 0$.

$$\mathcal{M}_\delta(\mu, \sigma) = \left\{ X \mid E(X) = \mu, \text{Var}(X) = \sigma^2, \hat{d}_W(F_X, F_0)^2 \leq \delta \right\},$$

where $\hat{d}_W$ is the “suitably” normalised Wasserstein distance of order 2 such that $0 \leq \delta \leq 1$. 
Wasserstein distance

\[ d_W(F, G)^2 = \int_{\mathbb{R}} (F(x) - G(x))^2 \, dx, \]

\[ = \int_0^1 (F^{-1}(u) - G^{-1}(u))^2 \, du, \]

\[ = \inf \left\{ E((X - Y)^2) \mid X \sim F, Y \sim G \right\}. \]

**Applications:** Optimal transport (1781), machine learning, robust statistics, neural networks, Wasserstein Auto-Encoders, image recognition...
Wasserstein distance

![Graph showing two normal distributions]

- Normal (0, 1)
- Normal (0, 1.5)
Wasserstein bound for ES
Wasserstein bound for ES

For a reference distribution $X_0 \sim F_0$ and tolerance distance $\delta \in [0, 1]$:

$$\left[ \inf_{X \in \mathcal{M}_\delta(\mu, \sigma)} \text{ES}_\alpha(X), \sup_{X \in \mathcal{M}_\delta(\mu, \sigma)} \text{ES}_\alpha(X) \right]$$

with uncertainty set

$$\mathcal{M}_\delta(\mu, \sigma) = \left\{ X \mid E(X) = \mu, \text{Var}(X) = \sigma^2, \hat{d}_W(F_X, F_0)^2 \leq \delta \right\}.$$
Wasserstein bound for ES

\( \text{ES}_\alpha(X) \) bounds with reference \( X_0 \) and tolerance distance \( \delta \):

\[
\begin{bmatrix}
\mu + \sigma c_{\alpha, \lambda}(X_0), & \mu + \sigma \frac{\alpha}{1-\alpha} + \lambda (\text{ES}_\alpha(X_0) - \mu) \\
\end{bmatrix}
\sqrt{\frac{\alpha}{1-\alpha} + \lambda (\text{ES}_\alpha(X_0) - \mu) + \lambda^2 \sigma^2},
\]

where \( \lambda \) is inverse proportional to \( \delta \):

- \( \delta = 0 \) corresponds to \( \lambda = +\infty \) \( \rightarrow \) \([\text{ES}_\alpha(X_0), \text{ES}_\alpha(X_0)]\).
- \( \delta = 1 \) corresponds to \( \lambda = 0 \) \( \rightarrow \) \([\mu, \mu + \sigma \sqrt{\frac{\alpha}{1-\alpha}}]\).
Upper bound for ES
Wasserstein upper bound for ES
Wasserstein upper bound for ES

\[ \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}} \]

\[ \delta \]

Risk Bounds under Uncertainty and Model Risk

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Wasserstein upper bound for ES
Wasserstein upper bound for ES
The distribution which attains the upper bound, has quantile function

\[ F^{-1}(u) = a + b \left( \frac{1}{1 - \alpha} \mathbb{1}_{(\alpha, 1]} + \lambda F_0^{-1}(u) \right), \]

where \( a, b \) are such that the mean and standard deviation constraint is fulfilled.
Wasserstein worst-case quantile for ES
Wasserstein worst-case quantile for ES
Wasserstein worst-case quantile for ES
Wasserstein worst-case quantile for ES
Lower and upper bound for ES
Wasserstein lower bound for ES
The quantile distributions which attain the $ES_{0.8}$ lower (dashed) and upper (solid) bounds:
Wasserstein bounds in practise
Recipe for deriving Wasserstein bounds

1. Choose reference distribution (empirical distribution) with sample mean and sample sd.
2. Choose tolerance distance $\delta \in [0, 1]$.
   - $\delta \ll 1$: low uncertainty;
   - $\delta \approx 1$: high uncertainty
3. Calculate $\lambda$ (inverse proportional to $\delta$).
4. Calculate bounds of $\text{ES}_{\alpha}$.
5. Calculate distribution which attains the bound.
How to choose the Wasserstein tolerance distance?

a) Distributional uncertainty, expert opinion

b) Model uncertainty, data driven uncertainty set
a) Distributional uncertainty

Assume we have uncertainty in some quantiles:

$$\{\text{VaR}_{\alpha_1}^1, \ldots, \text{VaR}_{\alpha_1}^{K_1}, \ldots, \text{VaR}_{\alpha_M}^1, \ldots, \text{VaR}_{\alpha_M}^{K_M} \}$$

Reason:

parameter uncertainty, expert opinions, additional data sources, ...
a) Distributional uncertainty

Assume we have uncertainty in some quantiles:

\[ \{ \text{VaR}_{\alpha_1}^1, \ldots, \text{VaR}_{\alpha_1}^{K_1}, \ldots, \text{VaR}_{\alpha_M}^1, \ldots, \text{VaR}_{\alpha_M}^{K_M} \} \]

**Reason:**

parameter uncertainty, expert opinions, additional data sources, ...

⇒ Choose \( \delta \) such that the uncertainty set \( \mathcal{M}_\delta(\mu, \sigma) \) is the smallest set containing all quantiles.
Wasserstein tolerance distance - distributional uncertainty

Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

<table>
<thead>
<tr>
<th>% uncertainty</th>
<th>$\delta$</th>
<th>$\text{ES}_{0.9}$</th>
<th>$\overline{\text{ES}}_{0.9}$</th>
<th>% bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.013</td>
<td>13.03</td>
<td>14.00</td>
<td>7%</td>
</tr>
<tr>
<td>3%</td>
<td>0.030</td>
<td>12.76</td>
<td>14.24</td>
<td>10%</td>
</tr>
<tr>
<td>5%</td>
<td>0.061</td>
<td>12.47</td>
<td>14.51</td>
<td>15%</td>
</tr>
<tr>
<td>10%</td>
<td>0.209</td>
<td>11.73</td>
<td>15.19</td>
<td>25%</td>
</tr>
</tbody>
</table>
Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

1% uncertainty in $\text{VaR}_{0.8}$, $\text{VaR}_{0.9}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\text{ES}_\alpha$</th>
<th>$\overline{\text{ES}}_\alpha$</th>
<th>% bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.01</td>
<td>13.03</td>
<td>14.00</td>
<td>7%</td>
</tr>
<tr>
<td>0.95</td>
<td>0.01</td>
<td>13.36</td>
<td>14.91</td>
<td>11%</td>
</tr>
<tr>
<td>0.97</td>
<td>0.01</td>
<td>13.52</td>
<td>15.61</td>
<td>14%</td>
</tr>
<tr>
<td>0.975</td>
<td>0.01</td>
<td>13.56</td>
<td>15.87</td>
<td>16%</td>
</tr>
</tbody>
</table>
b) Model uncertainty

- $F_0$ is the true unknown distribution.
- Let $F_N$ be the empirical distribution.
- Assume the sample mean and sd converge to the mean and sd of $F_0$.

Choose $\delta$ such that the true distribution lies in the uncertainty set with probability $1 - \beta$. That is

$$P \left( \hat{d}_W(F_N, F) \leq \delta \right) \geq 1 - \beta.$$
Assume that, for some $\alpha > 2$,

$$E(e^{X^\alpha}) < \infty.$$ 

Then,

$$\delta \approx \sqrt{\frac{\log(C/\beta)}{N}},$$

where $N$ the sample size and $C \approx 2(E(e^{X^\alpha}) + E(e^{(X/2)^\alpha}) - 1)$. 
Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>$10^6$</td>
<td>0.063</td>
</tr>
<tr>
<td>5%</td>
<td>$10^6$</td>
<td>0.062</td>
</tr>
<tr>
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<td>0.061</td>
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<tr>
<th>$\beta$</th>
<th>$N$</th>
<th>$\delta$</th>
<th>% ES$_{0.9}$ bounds</th>
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<tbody>
<tr>
<td>5%</td>
<td>$10^5$</td>
<td>0.066</td>
<td>15%</td>
</tr>
<tr>
<td>5%</td>
<td>$10^6$</td>
<td>0.020</td>
<td>8%</td>
</tr>
<tr>
<td>5%</td>
<td>$10^7$</td>
<td>0.007</td>
<td>5%</td>
</tr>
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Wasserstein bounds for VaR
Recall that

$$\text{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_u(X) du.$$  

The methodology for the ES Wasserstein bounds also apply to the RVaR.
Recall that

\[
RVaR_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_u(X) du.
\]

The methodology for the ES Wasserstein bounds also apply to the RVaR. Moreover, we have

\[
\lim_{\alpha' \uparrow \alpha} RVaR_{\alpha',\alpha} = \text{VaR}_\alpha.
\]

Thus, we obtain Wasserstein bounds for the VAR.
Recall that
\[
RVaR_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_u(X) du.
\]

The methodology for the ES Wasserstein bounds also apply to the RVaR. Moreover, we have
\[
\lim_{\alpha' \uparrow \alpha} RVaR_{\alpha',\alpha} = \text{VaR}_\alpha.
\]

Thus, we obtain Wasserstein bounds for the VAR.
⇒ Future work: assessment of numerical stability of VaR bounds.
Derived bounds for the ES under Wasserstein uncertainty.

Wasserstein uncertainty includes distribution with same mean and sd and which are close in the Wasserstein distance.

Bounds depend on the reference distribution.

Ways of choosing the Wasserstein tolerance distance, via model and distributional uncertainty.
1. Easily extendable to uncertainty in the mean and standard deviation, e.g. \((\mu, \sigma) \in [\mu, \mu'] \times [\sigma, \sigma']\)
1. Easily extendable to uncertainty in the mean and standard deviation, e.g. $(\mu, \sigma) \in [\underline{\mu}, \bar{\mu}] \times [\bar{\sigma}, \sigma]$

2. Applicable to any risk measure of the form:

$$\rho(X) = \int_{0}^{1} F_X^{-1}(u) \gamma(u) du,$$

for a density $\gamma$ on $[0, 1]$.

⇒ For example to any spectral risk measure.
3. Risk bounds for aggregate risks?

\[ \inf_{X \in \mathcal{M}} \rho \left( \sum_{i=i}^{d} X_i \right), \quad \sup_{X \in \mathcal{M}} \rho \left( \sum_{i=i}^{d} X_i \right). \]

a) non-linear aggregation \( g(X_1, \ldots, X_n) \)?

b) choice of \( \mathcal{M} \)?

c) Incorporating uncertainty in the marginals \( X_1, \ldots X_n \)?

d) Incorporating uncertainty in the dependence (copula)?
Thank you!
References


For $\delta \in [0, 1]$, set

$$
\varepsilon = 2\sigma^2 \delta \left(1 - \frac{\text{ES}_\alpha(X_0) - \mu}{\sigma \sqrt{\frac{\alpha}{1-\alpha}}} \right)
$$

Then, $\lambda \in (0, \infty)$ is the solution to

$$
\frac{\varepsilon}{2\sigma^2} = 1 - \frac{\text{ES}_\alpha(X_0) - \mu + \lambda\sigma^2}{\sigma \sqrt{\frac{\alpha}{1-\alpha}} + \lambda^2\sigma^2 + 2\lambda \left(\text{ES}_\alpha(X_0) - \mu\right)}.
$$
The non-normalised Wasserstein tolerance distance is given by

$$
\varepsilon = \max_{i=1,\ldots,M} \max_{k=1,\ldots,K} \int_{\alpha_i}^{1} \left( \text{VaR}_{\alpha_i}^k - F_0^{-1}(u) \right)^2 \, du.
$$