Economic Theory of Financial Markets

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Abstract

These are our notes from a lecture by Dr. Mario Wüthrich in spring 2009.

Contents

1 Utility Theory

1.1 Expected Utility and Risk Aversion

Let $(\Omega, \mathcal{F}, P)$ a probability space and $\mathcal{X} \subset \{X : \Omega \to \mathbb{R}\}$ possible choices of an economic agent, represented by random variables X.

Definition 1.1. A *preference order* on $\mathcal{X}$ is a relation $\succeq$ with the following properties:

1. (Completeness): For all $X, Y \in \mathcal{X}$ we have $X \succeq Y$ or $Y \succeq X$.
2. (Transitivity): For all $X, Y, Z \in \mathcal{X}$ we have $X \succeq Y, Y \succeq Z \Rightarrow X \succeq Z$.

Remark. If $X \succeq Y$ and $Y \succeq X$ we note $X \sim Y$. The economic agent is *indifferent* between X and Y.

Definition 1.2. A *numerical representation* for a preference order $\succeq$ on $\mathcal{X}$ is a function $U : \mathcal{X} \to \mathbb{R}$ such that $U(X) \geq U(Y)$ if and only if $X \succeq Y$.

Remark. A numerical representation is not unique.

Remark. Necessary and sufficient conditions on $\succeq$ for the existence of a numerical representation can be found in Theorem 2.6 of [FS04].

Let $I \subset \bar{\mathbb{R}}$ an interval such that $P(X \in I) = 1$ for all $X \in \mathcal{X}$. For a given map $u : I \to \mathbb{R}$ a numerical representation U can be constructed by

$$U(X) = \mathrm{E}[u(X)] = \int_I u(x)\mu_X(\mathrm{d}x) \tag{1}$$

with $\mu_X(B) = P(X \in B)$, the distribution of X

Definition 1.3. A numerical representation of the form (1) is called *von Neumann–Morgenstern utility.* If u is strictly increasing then u is called *utility function.*

Remark. In the two period model X represents the payout at time 1. The utility $U(X) = \mathrm{E}[u(X)]$ can be interpreted as the average happyness with the choice X at time 1.

Lemma 1.4. Any positive linear transformation v of the utility function u generates the same preference order.

$$v(x) = a + bu(x), \quad a, b \in \mathbb{R}, b > 0$$

Proof. Exercise. □

Assumption 1.5. We will always assume $\mathcal{X} \subset L^1(\Omega, \mathcal{F}, P)$ i.e. $\mathrm{E}[X] < \infty$ for all $X \in \mathcal{X}$.

Definition 1.6. Assume the financial agent has utility function u. The financial agent is called *risk averse* if $u(\mathrm{E}[X]) \geq \mathrm{E}[u(X)]$, *risk neutral* if $u(\mathrm{E}[X]) = \mathrm{E}[u(X)]$ and *risk seeking* if $u(\mathrm{E}[X]) \leq \mathrm{E}[u(X)]$.

Lemma 1.7. If a financial agent has strictly concave utility function u, then he is risk averse.

Proof. Jensens inequality. □

Remark. If X is not deterministic and u is strictly concave then $\mathrm{E}[u(X)] < u(\mathrm{E}[x])$.

Assumption 1.8.

- Assume u is three times differentiable on I, $u \in \mathcal{C}^3$
- Assume risk averse utility functions are strictly concave. $u'(x) > 0$, $u''(x) < 0$.

Example.

- *Exponential utility*

$$u(x) = -\frac{1}{\alpha}e^{-\alpha x}$$

  with $I = \mathbb{R}$ and $\alpha > 0$
  (Actuaries choice)

- *Power utility*

$$u(x) = \begin{cases} \frac{1}{1-\alpha}(x^{1-\alpha} - 1) & \text{for } \alpha \neq 1 \\ \log(x) & \text{for } \alpha = 1 \end{cases}$$

  with $I = \mathbb{R}_+$ and $\alpha > 0$
  (Economists choice)

Definition 1.9. The *absolute risk aversion* is given by

$$\rho_{\mathrm{ARA}}(x) = -\frac{u''(x)}{u'(x)}$$

The *relative risk aversion* is given by

$$\rho_{\mathrm{RRA}}(x) = -x\frac{u''(x)}{u'(x)}$$

Remark. Risk averse financial agents have $\rho_{\mathrm{ARA}}(x) > 0$ and $\rho_{\mathrm{RRA}}(x) > 0$.

Example.

- The exponential utility satisfies $\rho_{\mathrm{ARA}}(x) \equiv \alpha$.
- The power utility satisfies $\rho_{\mathrm{RRA}}(x) \equiv \alpha$.

1.2 Indifference Price

Assume a financial agent with utility function u and initial wealth $w \in \mathbb{R}$.

Definition 1.10. Let $X \in \mathcal{X}$. The *certainty eqivalent* for (X, u, w) is given by the number $x \in \mathbb{R}$ satisfying

$$u(w + x) = \mathrm{E}[u(w + X)]$$

or equivalently, using preference order notation

$$w + x \sim w + X$$

Remark. The certainty equivalent x is a function of X, u and w. $x = x(X, u, w)$.

Lemma 1.11. If u is risk averse then $\mathrm{E}[X] \geq x$.

Proof.

$$\begin{aligned} u(w + x) &= \mathrm{E}[u(w + X)] \\ &\leq u(\mathrm{E}[w + X)] \\ &= u(w + \mathrm{E}[X]) \end{aligned}$$

and since u is increasing

$$\begin{aligned} w + x &\leq w + \mathrm{E}[X] \\ x &\leq \mathrm{E}[X] \end{aligned}$$

□

Definition 1.12. A financial agent 1 with utility function u_1 is called *more risk averse* than agent 2 with utility function u_2 if

$$u_1^{-1}(\mathrm{E}[u_1(X)]) \leq u_2^{-1}(\mathrm{E}[u_2(X)])$$

for all $X \in \mathcal{X}$.

For the following we will identify the set $\mathcal{X}$ of random variables with the set $\mathcal{X}'$ of their probability distributions: $\mathcal{X}' = \{\mu_X(B) = P(X \in B) : X \in \mathcal{X}\} \subset \mathcal{M}_1(I)$

Assumption 1.13. $\mathcal{X} \subset \mathcal{M}_1(I)$ is convex and contains all point measures δ_y for $y \in I$.

For instance if $y \in I$, $z \in I$ then $\frac{1}{2}\delta_y + \frac{1}{2}\delta_z \in \mathcal{X}$.

Proposition 1.14. The following statements are equivalent:

1. Agent 1 is more risk averse than agent 2.

2. $\rho^{(1)}_{\text{ARA}}(y) \geq \rho^{(2)}_{\text{ARA}}(y)$ for all $y \in I$.

3. There is a strictly increasing concave function v such that $u_1(y) = v(u_2(y))$ for all $y \in I$.

Proof.

Statement (2) implies statement (3):

Let $z = u_2^{-1}$ and define $v = u_1 \circ u_2^{-1}$ we will show that v satisfies the conditions from statement (3).

Clearly $v(u_2(y)) = u_1(u_2^{-1}(u_2(y))) = u_1(y)$.

Since u_2 is strictly increasing, so is u_2^{-1} , and since u_1 is also strictly increasing, so is their composition v.

In order to proof concavity we look at the second derivative of v.

$$\begin{aligned}
v' &= (u_1 \circ z)' = (u_1' \circ z) \cdot z' = \frac{u_1'(z)}{u_2'(z)} > 0 \\
v'' &= \frac{u_1''(z)u_2'(z) - u_1'(z)u_2''(z)}{(u_2'(z))^2} z' \qquad\qquad \text{note: } z' = \frac{1}{u_2'} \\
&= \frac{u_1'(z)}{(u_2'(z))^2} \left(\frac{u_1''(z)}{u_1'(z)} - \frac{u_2''(z)}{u_2'(z)} \right) \\
&= \underbrace{\frac{u_1'(z)}{(u_2'(z))^2}}_{\geq 0} \underbrace{\left(-\rho^{(1)}_{\text{ARA}} + \rho^{(2)}_{\text{ARA}} \right)}_{\leq 0} \leq 0
\end{aligned}$$

by assumption of statement (2), thus proving concavity of v.

Statement (3) implies statement (2):

The function v given by statement (3) is assumed concave thus $v'' < 0$. The identity

$$v'' = \underbrace{\frac{u_1'(z)}{(u_2'(z))^2}}_{\geq 0} \left(-\rho^{(1)}_{\text{ARA}} + \rho^{(2)}_{\text{ARA}} \right)$$

proven above holds. Thus we have $\rho^{(1)}_{\text{ARA}} \geq \rho^{(2)}_{\text{ARA}}$.

An equivalent formulation of statement (1):

Agent 1 is more risk averse than agent 2

$$\begin{aligned}
&\Leftrightarrow \quad u_1^{-1}(\mathrm{E}[u_1(X)]) \leq u_2^{-1}(\mathrm{E}[u_2(X)]) && \text{for all } X \in \mathcal{X} \\
&\Leftrightarrow \quad \mathrm{E}[u_1(X)] \leq u_1(u_2^{-1}(\mathrm{E}[u_2(X)])) && \text{for all } X \in \mathcal{X} \\
&\Leftrightarrow \quad \mathrm{E}[u_1(X)] \leq v(\mathrm{E}[u_2(X)]) && \text{for all } X \in \mathcal{X}
\end{aligned}$$

Statement (3) implies statement (1):

Applying Jensens inequality on the convex function $(-v)$:

$$\mathrm{E}[-v(u_2(X))] \geq -v(\mathrm{E}[u_2(X)])$$

the equivalent formulation of (1) is proven.

$$\mathrm{E}[u_1(X)] = \mathrm{E}[v(u_2(X))] \leq v(\mathrm{E}[u_2(X)]).$$

Satement (1) implies statement (2):

We will proceed by a proof by contradiction. Assume (2) is false. There exists $y \in I$ such that $\rho_{\mathrm{ARA}}^{(1)}(y) < \rho_{\mathrm{ARA}}^{(2)}(y)$. Since these functions are continuous there exists an open neighbourhood $\theta \subset I$ of y such that $\rho_{\mathrm{ARA}}^{(1)}(t) < \rho_{\mathrm{ARA}}^{(2)}(t)$ for all $t \in \theta$.

Then v is strictly convex on θ.

Choose a distribution $\mu \in \mathcal{X}$ with support θ which is not concentrated in a single point. Let $Y \sim \mu$. By Jensens inequality

$$\mathrm{E}[u_1(Y)] = \mathrm{E}[v(u_2(Y))] > v(\mathrm{E}[u_2(Y)])$$

contradicting (1). □

Corollary 1.15. If $\rho_{\mathrm{ARA}}^{(1)} \geq \rho_{\mathrm{ARA}}^{(2)}$ then the certainity equivalents of the agents satisfy $x(X, u_1, w) \leq x(X, u_2, w)$.

"The more risk averse an agent is, the less he will pay for risk X."

Proof. In the corollary we assume statement (2) of the proposition.

Let $x_1 = x(X, u_1, w)$, $x_2 = x(X, u_2, w)$.

Using statement (1) we have

$$\begin{gathered}
u_1^{-1}(\mathrm{E}[u_1(X+w)]) \leq u_2^{-1}(\mathrm{E}[u_2(X+w)]) \\
x_1 + w \leq x_2 + w
\end{gathered}$$

□

Definition 1.16. Assume a financial agent with utility u and wealth $w > 0$. The *indifference price* for risk $X \in \mathcal{X}$ is the solution $\Pi = \Pi(X, u, w)$ of the equation

$$u(w) = \mathrm{E}[u(w + \Pi - X)] \,. \tag{2}$$

Remark. The certainty equivalent of $\Pi - X$ is zero.

Definition 1.17. The *risk premium* Π_r is given by $\Pi_\mathrm{r} = \Pi - \mathrm{E}[X]$.

Lemma 1.18. If u is risk averse then $\Pi_\mathrm{r} > 0$.

Proof. The claim follows directly from Jensens inequality:

$$u(w) = \mathrm{E}[u(w + \Pi - X)] < u(\mathrm{E}[w + \Pi - X]) = u(w + \Pi_r)$$

since u is strictly increasing this implies $0 < \Pi_r$. □

Theorem 1.19. If u is risk averse (i.e. concave) then the following statements are equivalent:

1. Π_r does not depend on w.
2. There are $A, B, \alpha \in \mathbb{R}, A > 0, \alpha > 0$ such that $u(x) = -Ae^{-\alpha x} + B$.

Remark. This theorem illustrates the weakness of exponential utility from the economists point of view: Intuitively the premium Π ought to decrease with increasing wealth w.

Proof. Assume statement (2). The definition of Π states

$$\begin{aligned}
u(w) &= \mathrm{E}[u(w + \Pi - X)] \\
-Ae^{-\alpha w} + B &= \mathrm{E}\Big[-Ae^{-\alpha(w+\Pi-X)} + B\Big] \\
&= \mathrm{E}\Big[-Ae^{-\alpha w}e^{-\alpha(\Pi-X)} + B\Big] \\
&= -Ae^{-\alpha w}e^{-\alpha\Pi}\,\mathrm{E}\Big[e^{-\alpha(-X)}\Big] + B \\
1 &= e^{-\alpha\Pi}\,\mathrm{E}\Big[e^{-\alpha(-X)}\Big] \\
e^{\alpha\Pi} &= \mathrm{E}\big[e^{\alpha X}\big] \\
\Pi &= \frac{1}{\alpha}\log(\mathrm{E}\big[e^{\alpha X}\big]) \\
\Pi_\mathrm{r} &= \frac{1}{\alpha}\log(\mathrm{E}\big[e^{\alpha X}\big]) - \mathrm{E}[X]
\end{aligned}$$

Thus prooving that Π_r does not depend on w.

Assume now, that Π_r does not depend on w. Thus $\frac{\mathrm{d}\Pi}{\mathrm{d}w} = 0$. If we derive equation (2) with respect to w we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}w} u(w) &= \frac{\mathrm{d}}{\mathrm{d}w} \mathrm{E}[u(w + \Pi - X)] \\
u'(w) &= \mathrm{E}\left[u'(w + \Pi - X)\left(1 + \underbrace{\frac{\mathrm{d}\Pi}{\mathrm{d}w}}_{=0}\right)\right] \qquad (3) \\
&= \mathrm{E}[u'(w + \Pi - X)]
\end{aligned}$$

This shows that Π is also the indifference price for the utility function $v = -u'$.

$$\begin{aligned}
w &= v^{-1}(\mathrm{E}[v(w + \Pi - X)]) \\
w &= u^{-1}(\mathrm{E}[u(w + \Pi - X)])
\end{aligned}$$

Thus u and v have the same risk aversion or more precicely: u is more risk averse than v and v is more risk averse than u. By proposition 1.14

$$\begin{aligned}
\rho^{(u)}_{\mathrm{ARA}}(x) &= \rho^{(v)}_{\mathrm{ARA}}(x) \qquad \text{for all } x \in I \\
-\frac{u''(x)}{u'(x)} &= -\frac{v''(x)}{v'(x)} = -\frac{u'''(x)}{u''(x)}
\end{aligned}$$

Now we observe the derivation of $\rho^{(u)}_{\mathrm{ARA}}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\rho^{(u)}_{\mathrm{ARA}}(x) &= -\frac{u'''(x)u'(x) - (u''(x))^2}{(u'(x))^2} \\
&= -\left(\frac{u'''(x)}{u'(x)} - \left(\frac{u''(x)}{u'(x)}\right)^2\right) \\
&= -\frac{u''(x)}{u'(x)} \underbrace{\left(\frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)}\right)}_{=0} \\
&= 0
\end{aligned}$$

Thus $\rho^{(u)}_{\mathrm{ARA}}$ needs to be constant. Let α denote its value.

$$\begin{aligned}
\rho^{(u)}_{\mathrm{ARA}}(x) &= \alpha \qquad \text{for all } x \in I \\
-\frac{u''(x)}{u'(x)} &= \alpha \\
u''(x) - \alpha u'(x) &= 0
\end{aligned}$$

The solution to this differential equation has the exponential form claimed. □

Theorem 1.20. Assume u is risk-averse. The following statements are equivalent:

1. The Risk Premium Π_r is decreasing in w.

2. $\rho_{\text{ARA}}(x)$ is decreasing in x.

Example. The power utility satisfies statement (2):

$$\rho_{\text{ARA}}(x) = \alpha x^{-1} \quad \text{with} \quad \alpha > 0$$

Proof. As in equation (3) we have

$$u'(w) = \mathrm{E}[u'(w + \Pi - X)]\,(1 + \Pi'(w))$$

therefore we have the following sequence of equivalences:

$$\begin{aligned}
\Pi'(w) \le 0 &\Leftrightarrow u'(w) \le \mathrm{E}[u'(w + \Pi - X)] \\
&\Leftrightarrow v(w) \ge \mathrm{E}[v(w + \Pi - X)] \\
&\Leftrightarrow \Pi(X, w, v) \ge \Pi(X, w, u) \\
&\Leftrightarrow \text{Agent } v \text{ is more risk averse than agent } u. \\
&\Leftrightarrow -\frac{u'''(x)}{u''(x)} = -\frac{v''(x)}{v'(x)} \ge \frac{u''(x)}{u'(x)} \\
&\Leftrightarrow \rho^{(v)}_{\text{ARA}}(x) \ge \rho^{(u)}_{\text{ARA}}(x)
\end{aligned}$$

From proposition 1.14 we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\rho_{\text{ARA}}(x) = \rho_{\text{ARA}}(x)\left(\frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)}\right) \le 0 \\
&\Leftrightarrow \frac{u'''(x)}{u''(x)} \le \frac{u''(x)}{u'(x)} \\
&\Leftrightarrow \Pi'(w) \le 0
\end{aligned}$$

□

1.3 Risk Exchange Economy

Assumption 1.21. Assume a finite probability space $|\Omega| < \infty$ and let $\mathcal{X}$ be the set of all $\mathcal{F}$–adapted positive random variables on $(\Omega, \mathcal{F}, P)$.

Assumption 1.22. Assume we have N financial agents and each holds a position which causes a payoff $Y_i \in \mathcal{X}$ at time 1. ($i \in \{1, \ldots, N\}$)

The market capitalization at time 1 is given by $Z = \sum Y_i$.

Assumption 1.23. Assume there is a financial pricing kernel $\varphi \in \mathcal{X}$ with $\mathrm{E}[\varphi] = 1$ such that the price of $X \in \mathcal{X}$ at time 0 is given by $\Pi(X) = \mathrm{E}[\varphi X]$

Remark. In economic literature φ is called the state price density, in financial literature φ is called financial pricing kernel and in actuarial literature φ is called state–price deflator.

Each agent is characterized by a utility function u_i, and he can be achieved by trading any $X_i \in \mathcal{X}$.

Agent i chooses his portfolio according to

$$\max_{X_i \in \mathcal{X}} \mathrm{E}[u_i(X_i)]$$

with constraint

$$\Pi(X_i) = \Pi(Y_i)$$

Proposition 1.24. (First Order Conditions) The optimal asset allocation for agent i is given by $u_i'(X_i) = \lambda_i \varphi$ for some $\lambda_i > 0$.

Proof. The Lagrangian for this optimization problem is

$$L = \mathrm{E}[u_i(X_i)] - \lambda_i(\Pi(X_i) - \Pi(Y_i)).$$

Perturb X_i by some εX with $X \in \mathcal{X}$ and $\varepsilon > 0$:

$$L(\varepsilon) = \mathrm{E}[u_i(X_i + \varepsilon X)] - \lambda_i(\underbrace{\Pi(X_i + \varepsilon X)}_{\mathrm{E}[\varphi(X_i+\varepsilon X)]} - \Pi(Y_i)).$$

In order for X_i to be optimal we have (for all $X \in \mathcal{X}$):

$$\begin{aligned} \frac{\mathrm{d}L(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0} &= [\mathrm{E}[u_i'(X_i + \varepsilon X)X] - \lambda_i \mathrm{E}[\varphi X]]\,|_{\varepsilon=0} \\ &\Rightarrow \mathrm{E}[u_i'(X_i)X] - \lambda_i \mathrm{E}[\varphi X] = 0 \quad \forall X \in \mathcal{X} \end{aligned}$$

Thus $u_i' = \lambda_i \varphi$ almost surely.

$$\mathrm{E}\left[\underbrace{u_i'(X_i)}_{>0}\right] = \lambda_i \underbrace{\mathrm{E}[\varphi]}_{=1} = \lambda_i > 0$$

□

Remark.

$$\mathrm{E}[u_i'(X_i)] = \lambda_i \mathrm{E}[\varphi] = \lambda_i$$

$$\varphi = \frac{u_i'(X_i)}{\mathrm{E}[u_i'(X_i)]}$$

Corollary 1.25. All optimal asset allocations X_i are commonotonic.

Proof. All agents are risk averse. $u_i'' < 0$. Therefore u_i' is strictly decreasing.

$$X_i = (u_i')^{-1}(\lambda_i \varphi)$$

For ω_1, ω_2 in Ω:

$$X_i(\omega_1) < X_i(\omega_2) \Leftrightarrow \varphi(\omega_1) < \varphi(\omega_2) \Leftrightarrow X_j(\omega_1) < X_j(\omega_2)$$

Thus X_i are commonotonic. □

Assumption 1.26. (Economic Principle)

We have a risk-exchange economy, i.e.

$$\sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} X_i$$

Remark. The value

$$Z = \sum_{i=1}^{N} Y_i$$

is called the market capitalization.

Theorem 1.27. All optimal asset allocations are commonotonic to Z.

Proof. X_i are commonotonic, thus $Z = \sum_i X_i = f(\varphi)$. □

Example. Exponential utilities, $\alpha_i > 0$.

$$u_i(x) = -\frac{1}{\alpha_i} e^{-\alpha_i x}$$

Note: we have a heterogeneous behaviour, since we may choose $\alpha_i \neq \alpha_k$ for $i \neq k$.

$$\begin{aligned} u_i'(x) &= e^{-\alpha_i x} \\ X_i &= -\frac{1}{\alpha_i} \log(\lambda_i \varphi) \end{aligned}$$

By Proposition 1.24 we have

$$\begin{aligned} X_i &= -\frac{1}{\alpha_1} \log \lambda_1 - \frac{1}{\alpha_1} \log \varphi \\ Z &= -\sum_{i=1}^{N} \frac{1}{\alpha_i} \log \lambda_i - \log \varphi \sum_{i=1}^{N} \frac{1}{\alpha_1} \end{aligned}$$

Define the market (or collective or aggregate) risk aversion as

$$\alpha = \left(\sum_{i=1}^{N} \frac{1}{\alpha_1} \right)^{-1}$$

and let

$$K = -\alpha \sum \frac{1}{\alpha_i} \log \lambda_i$$

then

$$\begin{aligned} Z &= \frac{1}{\alpha} K - \frac{1}{\alpha} \log \varphi \\ \varphi &= e^{K - \alpha Z} \end{aligned}$$

In fact, φ is chosen endogeneously. We normalize

$$\begin{aligned} 1 &= \mathrm{E}[\varphi] \\ 1 &= e^K \, \mathrm{E}\big[e^{-\alpha Z}\big] \\ K &= -\log \mathrm{E}\big[e^{-\alpha Z}\big] \\ \varphi &= \frac{e^{-\alpha Z}}{\mathrm{E}[e^{-\alpha Z}]} \\ \Pi(X) = \mathrm{E}[\varphi X] &= \frac{\mathrm{E}\big[e^{-\alpha Z} X\big]}{\mathrm{E}[e^{-\alpha Z}]} \end{aligned}$$

Escher Price for X, see [Büh80]

$$X_i = -\frac{1}{\alpha_i} \log \lambda_i - \frac{1}{\alpha_i} \log \frac{e^{\alpha Z}}{\mathrm{E}[e^{-\alpha Z}]} = \frac{\alpha}{\alpha_i} Z - \frac{1}{\alpha_i} \log \frac{\lambda_i}{\mathrm{E}[e^{-\alpha Z}]}$$

$$\Pi(Y_i) = \Pi(X_i) = \mathrm{E}[\varphi X_i] = \frac{\alpha}{\alpha_i} \, \mathrm{E}[\varphi Z] - \frac{1}{\alpha_i} \log \frac{\lambda_i}{\mathrm{E}[e^{-\alpha Z}]}$$

$$\Rightarrow -\frac{1}{\alpha_i} \log \frac{\lambda_i}{\mathrm{E}[e^{-\alpha Z}]} = \Pi(Y_i) - \frac{\alpha}{\alpha_i} \Pi(Z)$$

$$X_i = \frac{\alpha}{\alpha_i} (Z - \Pi(Z)) + \Pi(Y_i)$$

Remark. Other utility functions (such as power utility) do not allow for a closed form solution if financial agents are heterogeneous.

2 Mean–Variance Analysis

Remark. Boldface letters $\boldsymbol{x}$ name vectors in $\mathbb{R}^n$:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Boldface letters $\widetilde{\boldsymbol{x}}$ with a tilde name vectors in $\mathbb{R}^{n+1}$:

$$\widetilde{\boldsymbol{x}} = (x_0, x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1}$$

2.1 The Markovitz Model

Assume $n+1$ financial assets with prices at time 0 given by $s_i^{(0)}$ and prices at time 1 given by the random variables $S_i^{(1)}$. $(i = 0, \ldots, n)$

Assume a financial agent holds a portfolio $\widetilde{\boldsymbol{a}}$. His wealth at time 0 is given by

$$w_0 = \sum_{i=0}^{n} a_i s_i^{(0)} \qquad \text{(deterministic)}$$

and his wealth at time 1 by

$$W_1 = \sum_{i=0}^{n} a_i S_i^{(1)} \qquad \text{(random).}$$

Definition 2.1. The *return* of asset i is given by

$$R_i = \frac{S_i^{(1)}}{s_i^{(0)}} - 1$$

We denote the proportion of asset i in the initial capital by

$$x_i = \frac{a_i s_i}{w_0}.$$

The wealth at time 1 can then be written as

$$W_1 = \sum_{i=0}^{n} a_i S^{(1)} = w_0 + w_0 \sum_{i=0}^{n} x_i R_i.$$

Note that

$$\sum_{i=0}^{n} x_i = 1.$$

The goal for a financial agent is to choose an investment strategy $\widetilde{\boldsymbol{x}}$ such that the expected wealth $\mathrm{E}[W_1]$ is large and its variance $\mathrm{Var}[W_1]$ is small.

Assume that the financial agent is risk averse with utility function u. His asset allocation problem can then be formulated as

$$\begin{aligned} &\text{maximize } \mathrm{E}[u(W_1)] \\ &\text{subject to} \\ &\qquad \widetilde{\boldsymbol{a}} \in \mathbb{R}^{n+1} \\ &\qquad \sum_{i=0}^{n} a_i s_i^{(0)} = w_0 \\ &\qquad \mathrm{E}[W_1] = w_1 \end{aligned}$$

with a given expected wealth $w_1 \in \mathbb{R}$.

Define the expected return by

$$r = \frac{w_1}{w_0} - 1$$

The above problem can then be rewritten as

$$\begin{aligned} &\text{maximize } \mathrm{E}\Big[u(w_0(1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}))\Big] \\ &\text{subject to} \\ &\qquad \widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1} \\ &\qquad \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} = 1 \\ &\qquad \widetilde{\boldsymbol{x}}^\top \mathrm{E}\Big[\widetilde{\boldsymbol{R}}\Big] = r \end{aligned}$$

using matrix multiplication notation and the vector $\widetilde{\boldsymbol{e}} = (1, \ldots, 1)$.

Assumption 2.2. (Model Assumptions)

Asset 0 is a risk free asset: $R_0 = \mu_0 > 0$ is deterministic.

The returns of the other (risky) assets follow a multivariate normal distribution with expected returns $\boldsymbol{\mu}$ and a positive definite covariance matrix Σ.

$$\boldsymbol{R} = (R_1, \ldots, R_n) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

The utility function is given by

$$u(x) = -\frac{1}{\alpha} e^{-\alpha x}.$$

with a given constant $\alpha > 0$.

Lemma 2.3. Under these model assumptions the random variable $e^{-\alpha W_1}$ has a log-normal distribution with parameters $-\alpha w_0(1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}})$ and $\alpha^2 w_0^2 \boldsymbol{x}^\top \Sigma \boldsymbol{x}$.

Proof.

$$\boldsymbol{R} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \Rightarrow \boldsymbol{x}^\top \boldsymbol{R} \sim \mathcal{N}(\boldsymbol{x}^\top \boldsymbol{\mu}, \boldsymbol{x}^\top \Sigma \boldsymbol{x})$$

(not proven here)

$$\begin{aligned}
-\alpha W_1 &= -\alpha w_0 (1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}) \\
&= -\alpha w_0 (1 + x_0 \mu_0 + \boldsymbol{x}^\top \boldsymbol{R}) \\
\mathrm{E}[-\alpha W_1] &= -\alpha w_0 (1 + \widetilde{\boldsymbol{x}} \widetilde{\boldsymbol{\mu}}) \\
\mathrm{Var}[-\alpha W_1] &= \alpha^2 w_0^2 \boldsymbol{x}^\top \Sigma \boldsymbol{x}
\end{aligned}$$

□

Corollary 2.4.

$$\begin{aligned}
\mathrm{E}[u(W_1)] &= -\frac{1}{\alpha} \mathrm{E}\left[e^{-\alpha W_1}\right] \\
&= -\frac{1}{\alpha} \exp(-\alpha w_0 (1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}}) + \frac{1}{2} \alpha^2 w_0^2 \boldsymbol{x}^\top \Sigma \boldsymbol{x})
\end{aligned}$$

Proof. The first identity follows directly from the definition of u.

The expected value of a lognormal distributed random variable with parameters m and s^2 is given by $\exp(m + \frac{1}{2} s^2)$. As in the Lemma set $m = -\alpha w_0 (1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}})$ and $s = \alpha^2 w_0^2 \boldsymbol{x}^\top \Sigma \boldsymbol{x}$. □

The problem can be rewritten to

$$\begin{aligned}
&\text{maximize } -\frac{1}{\alpha} \exp(-\alpha w_0 (1 + \widetilde{\boldsymbol{x}}_r^\top \widetilde{\boldsymbol{\mu}}) + \frac{1}{2} \alpha^2 w_0^2 \boldsymbol{x}_r^\top \Sigma \boldsymbol{x}) \\
&\text{subject to} \\
&\qquad \widetilde{\boldsymbol{x}}_r \in \mathbb{R}^{n+1} \\
&\qquad \widetilde{\boldsymbol{x}}_r^\top \widetilde{\boldsymbol{e}} = 1 \\
&\qquad \widetilde{\boldsymbol{x}}_r^\top \mathrm{E}\left[\widetilde{\boldsymbol{R}}\right] = r
\end{aligned}$$

or equivalently, eliminating constant terms and factors from the target function

$$\begin{aligned}
&\text{minimize } \boldsymbol{x}_r^\top \Sigma \boldsymbol{x} \\
&\text{subject to} \\
&\qquad \widetilde{\boldsymbol{x}}_r \in \mathbb{R}^{n+1} \\
&\qquad \widetilde{\boldsymbol{x}}_r^\top \widetilde{\boldsymbol{e}} = 1 \\
&\qquad \widetilde{\boldsymbol{x}}_r^\top \mathrm{E}\left[\widetilde{\boldsymbol{R}}\right] = r
\end{aligned}$$

Definition 2.5. The above problem is called the *Markowitz Problem.*

It can be spelled out as “Achieve a given return r with minimal variance”.

2.2 Preliminaries

Definition 2.6. A matrix A is called *positive definite* if $\boldsymbol{x}^\top A\boldsymbol{x} > \mathbf{0}$ for any non-zero vector $\boldsymbol{x} \neq \mathbf{0}$.

Proposition 2.7. A posivie definite matrix has a positive definite inverse.

Proposition 2.8. If $\boldsymbol{X}$ is a random vector with covariance matrix V then

$$\mathrm{Cov}[A\boldsymbol{X} + \boldsymbol{b}, C\boldsymbol{X} + \boldsymbol{d}] = AVC^\top$$

Definition 2.9. Let $f : \mathbb{R}^n \to \mathbb{R}$ a sufficiently smooth function. The *gradient* of f at $\boldsymbol{x}$ is given by

$$\nabla f(\boldsymbol{x}) = \frac{\partial f}{\partial \boldsymbol{x}} = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \dots \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right).$$

Definition 2.10. The *Hessian* of a sufficiently smooth f is given by the matrix

$$\mathrm{H}f = \frac{\partial^2 f}{(\partial \boldsymbol{x})^2} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1\dots n}$$

Example. Let $A \in \mathbb{R}^{n\times n}$ and $f(\boldsymbol{x}) = \boldsymbol{x}^\top A\boldsymbol{x}$. Then

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= (A + A^\top)\boldsymbol{x} \\ \mathrm{H}f(\boldsymbol{x}) &= (A + A^\top) \end{aligned}$$

2.3 Optimization

Unconstraint Unconstraint local maxima of a sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}$ are found by solving the following system:

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= \mathbf{0} \\ \boldsymbol{z}^\top \mathrm{H}f(\boldsymbol{x})\boldsymbol{z} &< 0 \quad \forall \boldsymbol{z} \neq \mathbf{0} \end{aligned}$$

Equality constraint Maxima with equality constraints

$$\begin{aligned} &\text{maximize } f(\boldsymbol{x}) \\ &\text{subject to} \\ &\qquad g(\boldsymbol{x}) = a \end{aligned}$$

solve the Lagrange problem

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{x}} &= \mathbf{0} \\ \frac{\partial L}{\partial \lambda} &= 0 \end{aligned}$$

where the Lagrange function L is defined as

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda(g(\boldsymbol{x}) - a)$$

The new variable λ is called the Lagrange-factor.

The problem can be written as

$$\begin{aligned} \nabla f(\boldsymbol{x}) - \lambda \nabla g(\boldsymbol{x}) &= \mathbf{0} \\ -(g(\boldsymbol{x}) - a) &= \mathbf{0} \end{aligned}$$

In order to have sufficient conditions for a constraint maximum, a second order condition needs to be added.

Inequality Constraint The solution of the following problem

$$\begin{aligned} &\text{maximize } f(\boldsymbol{x}) \\ &\text{subject to} \\ &\qquad g(\boldsymbol{x}) \geq a \end{aligned}$$

satisfies these necessary conditions (Kuhn-Tucker-Problem):

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{x}} &= \mathbf{0} \\ \frac{\partial L}{\partial \lambda} &= 0 \\ \frac{\partial L}{\partial b} &= \lambda \\ (b - a)\lambda &= 0, \quad b \geq a \end{aligned}$$

where

$$L(\boldsymbol{x}, \lambda, b) = f(\boldsymbol{x}) - \lambda(g(\boldsymbol{x}) - b)$$

2.4 Mean–Variance Analysis without risk free assets

Assumption 2.11. (Model assumptions) There are n assets with returns $\boldsymbol{R} = (R_1, \ldots, R_n)$.

1. Let $\boldsymbol{\mu} = \mathrm{E}[\boldsymbol{R}]$. μ has at least one coordinate that differs from the others.
2. Let $V = \mathrm{Cov}[\boldsymbol{R}]$. V is positive definite.

Definition 2.12. $R_p(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{R}$ is the *portfolio return.*

Remark. The expected return is equal to $\mathrm{E}[R_p(\boldsymbol{x})] = \boldsymbol{x}^\top \boldsymbol{\mu}$. The variance of $R_p(\boldsymbol{x})$ is given by $\boldsymbol{x}^\top V \boldsymbol{x}$.

Definition 2.13. A vector $\boldsymbol{x}$ is called an *investment strategy* if $\boldsymbol{x}^\top \boldsymbol{e} = 1$. (Recall that $\boldsymbol{e} = (1, \ldots, 1)$.)

Definition 2.14. An investment strategy $\boldsymbol{z}$ is called *efficient* if there is no investment strategy $\boldsymbol{x}$ with

$$\mathrm{E}[R_p(\boldsymbol{x})] \geq \mathrm{E}[R_p(\boldsymbol{z})] \qquad \text{and} \qquad \mathrm{Var}[R_p(\boldsymbol{x})] < \mathrm{Var}[R_p(\boldsymbol{z})]$$

Our goal is to find efficient investment strategies. For a given portfolio return r_p we will study the two problems

$$\boldsymbol{x}_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{e} = 1 \\ \boldsymbol{x}^\top \boldsymbol{\mu} = r_p}}{\arg\min} \ \boldsymbol{x}^\top V \boldsymbol{x} \tag{4}$$

$$\boldsymbol{x}^+_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{e} = 1 \\ \boldsymbol{x}^\top \boldsymbol{\mu} \geq r_p}}{\arg\min} \ \boldsymbol{x}^\top V \boldsymbol{x} \tag{5}$$

Note that in (4) we want to achieve *exactly* the given return, while in (5) we want to achieve *at least* the given return.

In order to simplify further calculations we will consider the equivalent problems

$$\boldsymbol{x}_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{e} = 1 \\ \boldsymbol{x}^\top \boldsymbol{\mu} = r_p}}{\arg\max} \ -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} \tag{6}$$

$$\boldsymbol{x}^+_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{e} = 1 \\ \boldsymbol{x}^\top \boldsymbol{\mu} \geq r_p}}{\arg\max} \ -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} \tag{7}$$

Problem (7) has the Lagrange function

$$L(\boldsymbol{x}, \lambda_1, \lambda_2, r) = -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} - \lambda_1(\boldsymbol{x}^\top \boldsymbol{e} - 1) - \lambda_2(\boldsymbol{x}^\top \boldsymbol{\mu} - r)$$

which leads to the following necessary conditions (Kuhn-Tucker approach):

$$\frac{\partial L}{\partial \boldsymbol{x}} = -V\boldsymbol{x} - \lambda_1 \boldsymbol{e} - \lambda_2 \boldsymbol{\mu} = 0 \tag{8a}$$

$$\frac{\partial L}{\partial \lambda_1} = -(\boldsymbol{x}^\top \boldsymbol{e} - 1) = 0 \tag{8b}$$

$$\frac{\partial L}{\partial \lambda_2} = -(\boldsymbol{x}^\top \boldsymbol{\mu} - r) = 0 \tag{8c}$$

$$\frac{\partial L}{\partial r} = \lambda_2 \leq 0 \tag{8d}$$

$$(r - r_p)\lambda_2 = 0 \quad r \geq r_p \tag{8e}$$

First Step Solve the Lagrange problem with fixed r consisting of (8a)–(8c).

Condition (8a) implies

$$\begin{aligned} \boldsymbol{x} &= -V^{-1}(\lambda_1 \boldsymbol{e} + \lambda_2 \boldsymbol{\mu}) \\ &= -V^{-1} \begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \end{aligned} \tag{9}$$

Note that V^{-1} exists since we assumed V to be positive definite. Now find λ_1 and λ_2 using (8b) and (8c). Define the 2×2 matrix A by

$$A = \begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top \end{pmatrix} V^{-1} \begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix}. \tag{10}$$

The conditions (8b) and (8c) imply that

$$\begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} \boldsymbol{e}^\top \boldsymbol{x} \\ \boldsymbol{\mu}^\top \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 1 \\ r \end{pmatrix}.$$

Using (9) we obtain

$$\begin{aligned} \begin{pmatrix} 1 \\ r \end{pmatrix} &= \begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top \end{pmatrix} V^{-1} \begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} \\ &= A \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} \end{aligned} \tag{11}$$

Proposition 2.15. The matrix A from (10) is positive definite.

Proof. Let $z = (z_1, z_2) \in \mathbb{R}^2$, $z \neq (0,0)$. Define

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Using the definition of A we obtain

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underbrace{\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top \end{pmatrix}}_{\boldsymbol{y}^\top} V^{-1} \underbrace{\begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{\boldsymbol{y}} = \boldsymbol{y}^\top V^{-1} \boldsymbol{y}.$$

We assumed that not all components of $\boldsymbol{\mu}$ are equal, therefore $\boldsymbol{\mu}$ and $\boldsymbol{e}$ are not collinear. Thus if $z \neq (0,0)$ then $\boldsymbol{y} \neq \boldsymbol{0}$. Since V is positive definite, so is V^{-1}, and therefore $\boldsymbol{y}^\top V^{-1} \boldsymbol{y} > 0$. Therefore

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} > 0.$$

Since z is chosen arbitrarily this proves that A is positive definite. □

Equation (11) can be solved and we find λ_1 and λ_2,

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ r \end{pmatrix} \tag{12}$$

and by equation (9) the solution $\boldsymbol{x_r}$ for the Lagrange problem is found.

$$\boldsymbol{x}_r = V^{-1} \begin{pmatrix} \boldsymbol{e} & \boldsymbol{\mu} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ r \end{pmatrix}$$

Mean–Variance Diagram By solving the Lagrange Problem we have chosen the equality constraint: $\mathrm{E}[R_p(\boldsymbol{x}_r)] = \mathrm{E}\big[\boldsymbol{x}^\top \boldsymbol{R}\big] = \boldsymbol{x}^\top \boldsymbol{\mu} = r$.

The Variance is given by

$$\begin{aligned}
\mathrm{Var}[R_p(\boldsymbol{x}_r)] &= \boldsymbol{x}_r^\top V \boldsymbol{x} \\
&= \overbrace{\begin{pmatrix}1 & r\end{pmatrix} A^{-1} \begin{pmatrix}\boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top\end{pmatrix} V^{-1}}^{\boldsymbol{x}^\top} V \overbrace{V^{-1} \begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix} A^{-1} \begin{pmatrix}1 \\ r\end{pmatrix}}^{\boldsymbol{x}} \\
&\qquad\qquad\underbrace{\hphantom{\begin{pmatrix}\boldsymbol{e}^\top \\ \boldsymbol{\mu}^\top\end{pmatrix} V^{-1} V V^{-1} \begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix}}}_{A} \\
&= \begin{pmatrix}1 & r\end{pmatrix} A^{-1} \begin{pmatrix}1 \\ r\end{pmatrix}
\end{aligned}$$

Let $a = \boldsymbol{e}^\top V^{-1} \boldsymbol{e}$, $b = \boldsymbol{e}^\top V^{-1} \boldsymbol{\mu}$, and $c = \boldsymbol{\mu}^\top V^{-1} \boldsymbol{\mu}$. Then

$$A = \begin{pmatrix}a & b \\ b & c\end{pmatrix}, \quad A^{-1} = \frac{1}{ac - b^2} \begin{pmatrix}c & -b \\ -b & a\end{pmatrix}$$

and

$$\begin{aligned}
\mathrm{Var}[R_p(\boldsymbol{x}_r)] &= \begin{pmatrix}1 & r\end{pmatrix} A^{-1} \begin{pmatrix}1 \\ r\end{pmatrix} \\
&= \frac{c - 2rb + r^2 a}{ac - b^2}
\end{aligned}$$

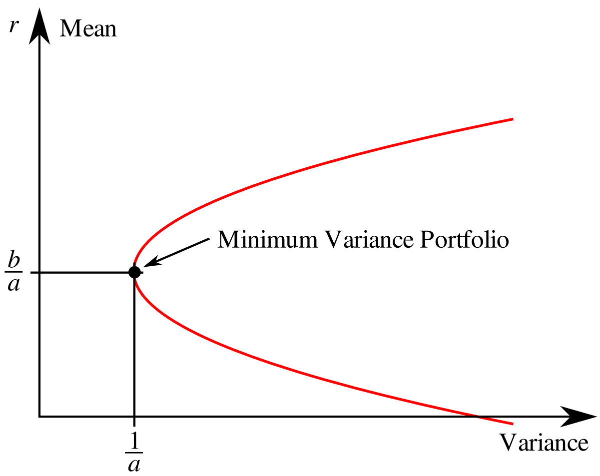

Figure 1: $\mathrm{Var}[R_p(\boldsymbol{x}_r)]$ as a function of r

If we chose $r = \frac{b}{a}$ then the portfolio $\boldsymbol{x}_r$ has minimal variance $\frac{1}{a}$.

Second Step Solve the Kuhn-Tucker problem (8a)–(8e).

We need to optimize the solution $\boldsymbol{x}_r$ in r such that $r \geq r_p$. Study three cases, for each case we must verify that the additional conditions (8d) and (8e) are fulfilled. (12) implies that $\lambda_2 = \frac{b-ra}{ac-b^2}$. Note that $ac - b^2 > 0$ since A is positive definite. The condition (8e) implies one of two possibilities: $r = r_p$ or $\lambda_2 = 0$.

1. $r_p < \frac{b}{a}$: The choice $r = r_p < \frac{b}{a}$ implies $\lambda_2 = \frac{b-r_p a}{ac-b^2} > 0$ which contradicts (8d).

   Therefore we have $0 = \lambda_2 = \frac{b-ra}{ac-b^2}$, thus $b - ra = 0$ and $r = \frac{b}{a}$. (Fig. 2)

2. $r_p = \frac{b}{a}$: Both choices ($r = r_p$ and $\lambda_2 = 0$) imply that $r = \frac{b}{a}$.

3. $r_p > \frac{b}{a}$: (8e) demands $r \geq r_p$. If we chose $r > r_p > \frac{b}{a}$ we have $\lambda_2 = \frac{b-ra}{ac-b^2} < 0$ and thus $(r - r_p)\lambda_2 < 0$ which contradicts (8e). Therefore we have $r = r_p$. (Fig. 3)

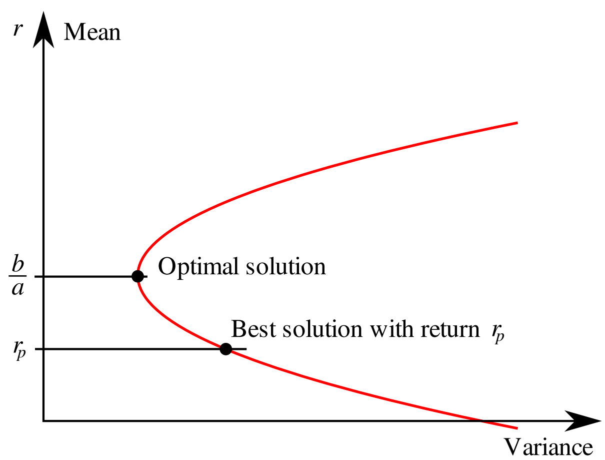

Figure 2: Solution for $r_p < \frac{b}{a}$

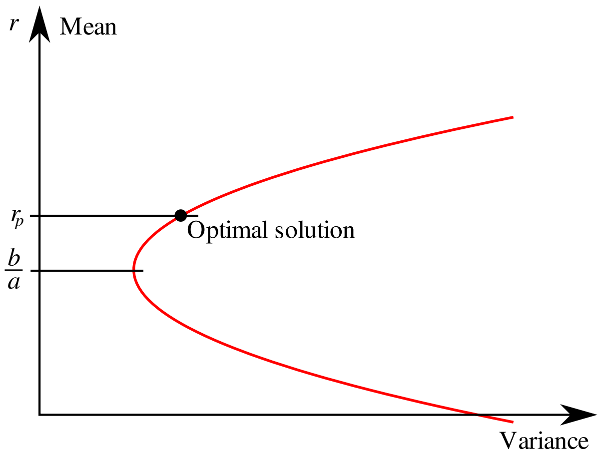

Figure 3: Solution for $r_p > \frac{b}{a}$

Definition 2.16. The solution $\boldsymbol{x}_r$ for the Lagrange problem (6) is called *minimum variance portfolio.*

Definition 2.17. The solution $\boldsymbol{x}^+_{r_p}$ for the Kuhn-Tucker problem (7) is called *efficient investment strategy.*

Definition 2.18. Let $r_{\text{gmv}} = \frac{b}{a}$. The *global minimum variance portfolio* $\boldsymbol{x}_{\text{gmv}}$ is given by $\boldsymbol{x}_{\text{gmv}} = \boldsymbol{x}_{r_{\text{gmv}}}$. Note that

$$\boldsymbol{x}_{\text{gmv}} = \frac{1}{\boldsymbol{e}^\top V^{-1}\boldsymbol{e}} V^{-1}\boldsymbol{e}$$

In this past section we have already proven the following theorem and its first corollary.

Theorem 2.19. Under the model assumptions 2.11 the solution to (7) is

$$\begin{aligned}\boldsymbol{x}^+_{r_p} &= \begin{cases} V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix} A^{-1}\begin{pmatrix}1\\ r_p\end{pmatrix} & \text{if } r_p \geq r_{\text{gmv}} \\ \boldsymbol{x}_{\text{gmv}} & \text{otherwise}\end{cases} \\ &= V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix} A^{-1}\begin{pmatrix}1\\ \max(r_p, r_{\text{gmv}})\end{pmatrix}.\end{aligned}$$

Corollary 2.20.

$$\boldsymbol{x}_r = V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix} A^{-1}\begin{pmatrix}1\\ r\end{pmatrix}$$

Example (Exercise)**.** Let $n = 3$, $\mu = (0.08, 0.04, 0.045)$ and

$$V = \begin{pmatrix}0.0025 & 0.0005 & 0.0010\\ 0.0005 & 0.0004 & 0.0006\\ 0.0010 & 0.0006 & 0.0010\end{pmatrix}$$

Find $r_{\text{gmv}} = 0.0395$.

Corollary 2.21. Every minimum variance portfolio is a linear combination of the two minimum variance portfolios $\boldsymbol{x}_{\text{gmv}}$ and $\boldsymbol{x}^{(0)}$ where

$$\boldsymbol{x}^{(0)} = \frac{1}{\boldsymbol{e}^\top V^{-1}\boldsymbol{\mu}} V^{-1}\boldsymbol{\mu}.$$

Every linear combination of $\boldsymbol{x}_{\text{gmv}}$ and $\boldsymbol{x}^{(0)}$ is a minimum variance portfolio.

Proof. Recall that

$$\boldsymbol{x}_r = V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix}\begin{pmatrix}-\lambda_1 \\ -\lambda_2\end{pmatrix}.$$

Let $r^{(0)} = \frac{c}{b}$. If we choose $r = r^{(0)}$ then (12) implies that $\lambda_1 = 0, \lambda_2 = \frac{-1}{\boldsymbol{e}^\top V^{-1}\boldsymbol{\mu}}$.

$$\begin{aligned}\boldsymbol{x}_{r^{(0)}} &= V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix}\begin{pmatrix}0 \\ \frac{1}{\boldsymbol{e}^\top V^{-1}\boldsymbol{\mu}}\end{pmatrix} \\ &= V^{-1}\boldsymbol{\mu}\frac{1}{\boldsymbol{e}^\top V^{-1}\boldsymbol{\mu}} \\ &= \boldsymbol{x}^{(0)},\end{aligned}$$

which shows that $\boldsymbol{x}^{(0)}$ is indeed a minimum variance portfolio.

Since A is positive definite, its columns are linearly independent and therefore $\frac{b}{a} \neq \frac{c}{b}$, thus $r_{\text{gmv}} \neq r^{(0)}$ and $\boldsymbol{x}_{\text{gmv}} \neq \boldsymbol{x}^{(0)}$.

We will now prove the first statement from the corollary, that every minimum variance portfolio is a linear combination of $\boldsymbol{x}_{\text{gmv}}$ and $\boldsymbol{x}^{(0)}$. Choose any return $r > 0$ and its corresponding minimum variance potfolio $\boldsymbol{x}_r$. Since $r_{\text{gmv}} \neq r^{(0)}$ there exists $\alpha \in \mathbb{R}$ such that $r = \alpha r^{(0)} + (1-\alpha)r_{\text{gmv}}$.

The minimum variance portfolio with return r is

$$\begin{aligned}\boldsymbol{x}_r &= V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix}A^{-1}\begin{pmatrix}1 \\ r\end{pmatrix} \\ &= V^{-1}\begin{pmatrix}\boldsymbol{e} & \boldsymbol{\mu}\end{pmatrix}A^{-1}\begin{pmatrix}\alpha + (1-\alpha) \\ \alpha r^{(0)} + (1-\alpha)r_{\text{gmv}}\end{pmatrix} \\ &= \alpha\boldsymbol{x}^{(0)} + (1-\alpha)\boldsymbol{x}_{\text{gmv}}\end{aligned}$$

which is indeed a linear combination of $\boldsymbol{x}^{(0)}$ and $\boldsymbol{x}_{\text{gmv}}$.

In order to prove the second statement chose $\alpha \in \mathbb{R}$ and let $\boldsymbol{x} = \alpha\boldsymbol{x}^{(0)} + (1-\alpha)\boldsymbol{x}_{\text{gmv}}$. We need to prove that this is indeed a minimum variance portfolio.

Let $r = \alpha r^{(0)} + (1-\alpha)r_{\text{gmv}}$ and observe (like above) that $\boldsymbol{x} = \boldsymbol{x}_r$ which proves that $\boldsymbol{x}$ is a minimum variance portfolio. □

Definition 2.22. Two investment strategies $\boldsymbol{x}$ and $\boldsymbol{y}$ are *orthogonal* if $\text{Cov}[R_p(\boldsymbol{x}), R_p(\boldsymbol{y})] = 0$.

Note that $\text{Cov}[R_p(\boldsymbol{x}), R_p(\boldsymbol{y})] = \boldsymbol{x}^\top V\boldsymbol{y}$.

Proposition 2.23. For every minimum variance portfolio $\boldsymbol{x}_r$ there exists an unique orthogonal minimum variance portfolio $\boldsymbol{x}_r^\perp$. Moreover $\boldsymbol{x}_r^\perp = \boldsymbol{x}_{r^\perp}$ where

$$r^\perp = \frac{c - br}{b - ar}.$$

Proof. Let $\lambda_1^\perp, \lambda_2^\perp$ be the solution to the Lagrange problem with expected return $r^\perp$.

$$\begin{aligned}\boldsymbol{x}_r &= -\lambda_1 V^{-1}\boldsymbol{e} - \lambda_2 V^{-1}\boldsymbol{\mu}\\ \boldsymbol{x}_{r^\perp} &= -\lambda_1^\perp V^{-1}\boldsymbol{e} - \lambda_2^\perp V^{-1}\boldsymbol{\mu}\end{aligned}$$

The covariance of the returns of these portfolios is

$$\begin{aligned}\mathrm{Cov}[R_p(\boldsymbol{x}_{r^\perp}), R_p(\boldsymbol{x}_r)] &= \boldsymbol{x}_{r^\perp} V \boldsymbol{x}_r\\ &= (-\lambda_1^\perp V^{-1}\boldsymbol{e} - \lambda_2^\perp V^{-1}\boldsymbol{\mu})^\top V(-\lambda_1 V^{-1}\boldsymbol{e} - \lambda_2 V^{-1}\boldsymbol{\mu})\\ &= (-\lambda_1^\perp \boldsymbol{e} - \lambda_2^\perp \boldsymbol{\mu})^\top V^{-1}(-\lambda_1 \boldsymbol{e} - \lambda_2 \boldsymbol{\mu})\\ &= \begin{pmatrix}\lambda_1^\perp & \lambda_2^\perp\end{pmatrix} A \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\\ &= \underbrace{\begin{pmatrix}1 & r^\perp\end{pmatrix} A^{-1}}_{\begin{pmatrix}\lambda_1^\perp & \lambda_2^\perp\end{pmatrix}} A \underbrace{A^{-1}\begin{pmatrix}1\\ r\end{pmatrix}}_{\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}}\\ &= \begin{pmatrix}1 & r^\perp\end{pmatrix} A^{-1}\begin{pmatrix}1\\ r\end{pmatrix}\\ &= \frac{1}{ac - b^2}\begin{pmatrix}1 & r^\perp\end{pmatrix}\begin{pmatrix}c - rb\\ ar - b\end{pmatrix}\\ &= \frac{c - rb - r^\perp b + arr^\perp}{ac - b^2}\end{aligned}$$

Until now we have used nothing about $r^\perp$. The above shows that

$$\mathrm{Cov}[R_p(\boldsymbol{x}_{r^\perp}), R_p(\boldsymbol{x}_r)] = 0 \quad \text{if and only if} \quad r^\perp = \frac{c - rb}{b - ar},$$

which proves the existence and uniqueness of the orthogonal minimum variance portfolio. □

Theorem 2.24. Assume $\boldsymbol{x}_r$ is a minimum variance portfolio for $r \neq \frac{b}{a}$ and $\boldsymbol{x}_{r^\perp}$ the corresponding orthogonal minimum variance portfolio.

Choose an arbitrary investment strategy $\boldsymbol{x} \in \mathbb{R}^n$ with expected investment return $r_{\boldsymbol{x}} = \boldsymbol{x}^\top \boldsymbol{\mu}$ and variance $\sigma_{\boldsymbol{x}}^2 = \boldsymbol{x}^\top V \boldsymbol{x}$. Then

$$r_{\boldsymbol{x}} - r^\perp = \beta_{\boldsymbol{x},r}(r - r^\perp)$$

where

$$\beta_{\boldsymbol{x},r} = \frac{\mathrm{Cov}[R_p(\boldsymbol{x}), R_p(\boldsymbol{x}_r)]}{\mathrm{Var}[R_p(\boldsymbol{x}_r)]}$$

Proof. Let $\boldsymbol{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ the vector with a single 1 in the j^{th} position.

$$\begin{aligned}
\operatorname{Cov}[R_j, R_p(\boldsymbol{x}_r)] &= \boldsymbol{e}_j^\top V \boldsymbol{x}_r \\
&= \boldsymbol{e}_j^\top V(-\lambda_1 V^{-1}\boldsymbol{e} - \lambda_2 V^{-1}\boldsymbol{\mu}) \\
&= \boldsymbol{e}_j^\top (-\lambda_1 \boldsymbol{e} - \lambda_2 \boldsymbol{\mu}) \\
&= -\lambda_1 - \lambda_2 \mu_j \\
&= \begin{pmatrix} 1 & \mu_j \end{pmatrix} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mu_j \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ r \end{pmatrix}
\end{aligned}$$

By definition of orthogonality we have

$$0 = \operatorname{Cov}[R_p(\boldsymbol{x}_{r^\perp}), R_p(\boldsymbol{x}_r)] = \boldsymbol{x}_{r^\perp}^\top V \boldsymbol{x}_r = \begin{pmatrix} 1 & r^\perp \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$\begin{aligned}
\operatorname{Cov}[R_j, R_p(\boldsymbol{x}_r)] &= \operatorname{Cov}[R_j, R_p(\boldsymbol{x}_r)] - \underbrace{\operatorname{Cov}[R_p(\boldsymbol{x}_{r^\perp}), R_p(\boldsymbol{x}_r)]}_{=0} \\
&= \begin{pmatrix} 0 & \mu_j - r^\perp \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ r \end{pmatrix} \\
&= \frac{(\mu_j - r^\perp)(-b + ar)}{ac - b^2}
\end{aligned}$$

Using the fact, that $R_p(\boldsymbol{x}) = \sum_j x_j R_j$ and linearity of covariance in its first argument as well as $\sum_j x_j = 1$, we obtain

$$\operatorname{Cov}[R_p(\boldsymbol{x}), R_p(\boldsymbol{x}_r)] = \sum_j x_j \operatorname{Cov}[R_j, R_p(\boldsymbol{x}_r)] = \frac{(\boldsymbol{x}^\top \boldsymbol{\mu} - r^\perp)(-b + ar)}{ac - b^2} \tag{13}$$

If we set $\boldsymbol{x} = \boldsymbol{x_r}$ in the above equation, we obtain

$$\operatorname{Var}[R_p(\boldsymbol{x}_r)] = \operatorname{Cov}[R_p(\boldsymbol{x}_r), R_p(\boldsymbol{x}_r)] = \frac{(\boldsymbol{x}_r^\top \boldsymbol{\mu} - r^\perp)(-b + ar)}{ac - b^2} = \frac{(r - r^\perp)(-b + ar)}{ac - b^2} \tag{14}$$

Equations (13) and (14) imply that

$$\frac{\operatorname{Cov}[R_p(\boldsymbol{x}), R_p(\boldsymbol{x}_r)]}{\boldsymbol{x}^\top \boldsymbol{\mu} - r^\perp} = \frac{\operatorname{Var}[R_p(\boldsymbol{x}_r)]}{r - r^\perp}$$

which proves the theorem. □

2.5 Mean–Variance Analysis with risk free assets

Assumption 2.25. Assume we have exactly one risk free asset with deterministic return $R_0 = \mu_0$ and n risky assets with returns $\boldsymbol{R}$ such that

1. $\mu_j = \mathrm{E}[R_j] \neq \mu_0$ for at least one j.
2. $V = \mathrm{Var}[\boldsymbol{R}]$ is positive definite.

We reformulate problem (7) in order to take into account the risk free asset.

$$\widetilde{\boldsymbol{x}}^+_{r_p} = \underset{\substack{\widetilde{\boldsymbol{x}} \in \mathbb{R}^n \\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} = 1 \\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq r_p}}{\arg\max} \; -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x}$$

The portfolio return can be reformulated in terms of excess return $\widetilde{\boldsymbol{R}}^e$:

$$R_p(\widetilde{\boldsymbol{x}}) = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}} = \sum_{i=0}^{n} x_i R_i = \sum_{i=0}^{n} x_i \underbrace{(R_i - \mu_0)}_{R_i^e} + \sum_{i=0}^{n} x_i \mu_0 = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}^e + \mu_0$$

The problems (6) and (7) become now

$$\boldsymbol{x}_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{\mu}^e = r_p^e}}{\arg\max} \; -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} \tag{15}$$

$$x_{r_p,0} = 1 - \boldsymbol{e}^\top \boldsymbol{x}_{r_p}$$

$$\boldsymbol{x}^+_{r_p} = \underset{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \boldsymbol{x}^\top \boldsymbol{\mu}^{\boldsymbol{e}} \geq r_p^e}}{\arg\max} \; -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} \tag{16}$$

$$x^+_{r_p,0} = 1 - \boldsymbol{e}^\top \boldsymbol{x}^+_{r_p}$$

where

$$r_p^e = r_p - \mu_0 \quad \text{and} \quad \boldsymbol{\mu}^e = \mathrm{E}[\boldsymbol{R}^e] = \boldsymbol{\mu} - \mu_0 \boldsymbol{e}$$

The Kuhn-Tucker conditions for (16) are

$$L(\boldsymbol{x}, \lambda, r^e) = -\tfrac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} - \lambda(\boldsymbol{x}^\top \boldsymbol{\mu}^e - r^e)$$

$$\frac{\partial L}{\partial \boldsymbol{x}} = -V\boldsymbol{x} - \lambda \boldsymbol{\mu}^e = 0 \tag{17a}$$

$$\frac{\partial L}{\partial \lambda} = -(\boldsymbol{x}^\top \boldsymbol{\mu}^e - r^e) = 0 \tag{17b}$$

$$\frac{\partial L}{\partial r^e} = \lambda \leq 0 \tag{17c}$$

$$(r^e - r_p^e)\lambda = 0 \quad r^e \geq r_p^e \tag{17d}$$

Equations (17a) and (17b) imply

$$\boldsymbol{x} = -\lambda V^{-1} \boldsymbol{\mu}^e$$

$$\boldsymbol{x}^\top \boldsymbol{\mu}^e = -\lambda \boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e = r^e$$

$$-\lambda = \frac{r_e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e}$$

So we can write the solution to the Lagrange problem (15)

$$\boldsymbol{x}_{r^e} = \frac{r_e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e} V^{-1} \boldsymbol{\mu}^e$$

Mean-Variance Diagram

$$\mathrm{Var}[R_p(\boldsymbol{x}_{r^e})] = \boldsymbol{x}_{r^e}^\top V \boldsymbol{x}_{r^e} = \frac{(r^e)^2}{(\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e)^2} \boldsymbol{\mu}^{e\top} V^{-1} V V^{-1} \boldsymbol{\mu}^e = \frac{(r^e)^2}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e}$$

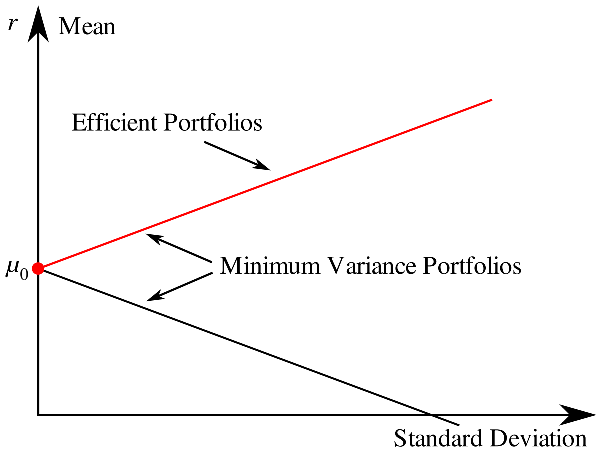

Figure 4: $\sqrt{\mathrm{Var}[R_p(\boldsymbol{x}_r)]}$ as a function of r

Theorem 2.26. Under assumptions 2.25 the solution to problem (16) is

$$\boldsymbol{x}_{r_p}^+ = \frac{\max(0, r_p^e)}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e} V^{-1} \boldsymbol{\mu}^e.$$

Corollary 2.27. The solution to problem (15) is

$$\boldsymbol{x}_{r^e}^+ = \frac{r^e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e} V^{-1} \boldsymbol{\mu}^e.$$

Proof. Theorem and Corollary are consequences from Kuhn-Tucker conditions (17a) – (17d). □

Definition 2.28. A minimum variance portfolio $\boldsymbol{x}_{\tan}$ is called *tangent portfolio* if $\boldsymbol{x}^\top \boldsymbol{e} = 1$.

(i.e. if $\widetilde{\boldsymbol{x}}_{\tan}$ invests only in risky assets.)

Proposition 2.29. There exists an unique tangential portfolio. Its excess return is

$$r_{\tan}^e = \frac{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{e}} \tag{18}$$

Proof. A minimum variance portfolio with given excess return r^e has the form

$$\boldsymbol{x}_{r^e} = \frac{r^e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e} V^{-1} \boldsymbol{\mu}^e$$

Thus

$$\boldsymbol{x}_{r^e}^\top \boldsymbol{e} = \frac{r^e}{\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{\mu}^e} \boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{e}$$

The equation $\boldsymbol{x}_{r^e}^\top \boldsymbol{e} = 1$ has the unique solution $r^e = r^e_{\tan}$. □

Remark. Practitionners would expect that $r_{\text{gmv}} > \mu_0$ since the latter return is achieved without any risk. Proposition 2.29 gives us the equivalence $r_{\text{gmv}} > \mu_0 \Leftrightarrow r^e_{\tan} > 0$.

Indeed: Since the numerator in (18) is positive, r^e has the same sign as

$$\boldsymbol{\mu}^{e\top} V^{-1} \boldsymbol{e} = (\boldsymbol{\mu} - \mu_0 \boldsymbol{e})^\top V^{-1} \boldsymbol{e} = b - \mu_0 a = a(r_{\text{gmv}} - \mu_0)$$

When estimating model parameters $\widetilde{\boldsymbol{\mu}}$ and V, make sure that this condition is fulfilled.

Theorem 2.30. Assume $\boldsymbol{x}_{r^e}$ is a minimum variance portfolio for $r^e \neq 0$, let $\widetilde{\boldsymbol{x}}$ an arbitrary portfolio with expected return $r_{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} = \mu_0 + \boldsymbol{x}^\top \boldsymbol{\mu}^e$. Then for $r = \mu_0 + r^e$ the following holds:

$$r_{\boldsymbol{x}} - \mu_0 = \beta_{\boldsymbol{x},r}(r - \mu_0) \text{ where } \beta_{\boldsymbol{x},r} = \frac{\operatorname{Cov}\left[\boldsymbol{x}^\top \boldsymbol{R}, \boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right]}{\operatorname{Var}\left[\boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right]}$$

Proof.

$$\begin{aligned}
\operatorname{Cov}[R_i, \boldsymbol{x}_{r^e} \boldsymbol{R}] &= \boldsymbol{e}_i^\top V \boldsymbol{x}_{r^e} \\
&= \boldsymbol{e}_i^\top V \left(\frac{r^e}{\boldsymbol{\mu}^\top V^{-1} \boldsymbol{\mu}^e} V^{-1} \boldsymbol{\mu}^e \right) \\
&= \frac{r^e}{\boldsymbol{\mu}^\top V^{-1} \boldsymbol{\mu}^e} \boldsymbol{e}_i^\top \boldsymbol{\mu}^{\boldsymbol{e}} \\
&= \frac{\operatorname{Var}\left[\boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right]}{r^e} \mu_i^e
\end{aligned}$$

Taking a linear combination with coefficients x_i of the above we obtain

$$\operatorname{Cov}\left[\boldsymbol{x}^\top \boldsymbol{R}, \boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right] = \frac{\operatorname{Var}\left[\boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right]}{r^e} \boldsymbol{x}^\top \boldsymbol{\mu}^e$$

□

Remark. The coefficient $\beta_{\boldsymbol{x},r}$ can be written as

$$\beta_{\boldsymbol{x},r} = \operatorname{Cor}\left[\boldsymbol{x}^\top \boldsymbol{R}, \boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right] \sqrt{\frac{\operatorname{Var}[\boldsymbol{x}^\top \boldsymbol{R}]}{\operatorname{Var}\left[\boldsymbol{x}_{r^e}^\top \boldsymbol{R}\right]}}$$

Remark. The portfolio $\boldsymbol{x}_{r^e}$ plays the role of a market portfolio. If the correlation is small then it is used for diversification. Due to the demand of such portfolios they are expensive, i.e. their excess return r^e is small.

Remark (Practical Remarks)**.**

1. There are often additional constraints to the optimization problem (6) such as e.g. a maximum of 30% of assets in foreign currency

$$\sum_{i \in \{\text{foreign}\}} x_i \leq 0.3$$

   or a no-short-selling constriaint

$$x_i \geq 0$$

2. The mean return vector $\boldsymbol{\mu}$ has to be estimated. An estimation based on historical data is often too naïve.

   The covariance matrix V can often be estimated with more confidence. Theorem 2.30 can then be used to estimate $\boldsymbol{\mu}$.

3. Asset and Liability management (ALM). Need to replicate a liability portfolio $\widetilde{\boldsymbol{y}}$ e.g. payouts of a life insurance using a deterministic life table or predicted payouts based on claim triangles in non-life insurance.

   This leads to the following problem. Note that not the variance of the return is minimized but the variance of the difference to the return of the given liabilites.

$$\boldsymbol{x}_{r_p}^{+} = \underset{\substack{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1} \\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} = 1 \\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq \widetilde{\boldsymbol{y}}^\top \widetilde{\boldsymbol{\mu}} + r_+}}{\arg\max} \quad -\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{y})^\top V(\boldsymbol{x} - \boldsymbol{y})$$

   where r_+ is an additional profit.

   As before we solve this problem using Kuhn-Tucker.

$$L(\widetilde{\boldsymbol{x}}, \lambda_1, \lambda_2, r) = -\frac{1}{2}\boldsymbol{x}^\top V \boldsymbol{x} + \boldsymbol{x}^\top V \boldsymbol{y} - \lambda_1(\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} - 1) - \lambda_2(\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} - (\widetilde{\boldsymbol{y}}^\top \widetilde{\boldsymbol{\mu}} + r))$$

$$\frac{\partial L}{\partial \boldsymbol{x}} = -V(\boldsymbol{x} - \boldsymbol{y}) - \lambda_1 \boldsymbol{e} - \lambda_2 \boldsymbol{\mu} = 0 \tag{19a}$$

$$\frac{\partial L}{x_0} = -\lambda_1 - \lambda_2 \mu_0 = 0 \tag{19b}$$

$$\frac{\partial L}{\partial \lambda_1} = -(\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} - 1) = 0 \tag{19c}$$

$$\frac{\partial L}{\partial \lambda_2} = -(\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} - (\widetilde{\boldsymbol{y}}^\top \widetilde{\boldsymbol{\mu}} + r)) = 0 \tag{19d}$$

$$\frac{\partial L}{\partial r} = \lambda_2 \leq 0 \tag{19e}$$

$$(r - r_+)\lambda_2 = 0 \quad r \geq r_+ \tag{19f}$$

Equations (19a) and (19b) imply

$$\boldsymbol{x} = \boldsymbol{y} - \lambda_2 V^{-1}\boldsymbol{\mu}^e$$

From (19d) we get

$$\widetilde{\boldsymbol{y}}^\top \widetilde{\boldsymbol{\mu}} + r = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} = \mu_0 + \boldsymbol{x}^\top \boldsymbol{\mu}^e = \mu_0 + \boldsymbol{y}^\top \boldsymbol{\mu}^e - \lambda_2 \boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{\mu}^e$$

therefore

$$-\lambda_2 = \frac{r}{\boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{\mu}^e}$$

Finally, using (19e) and (19f), we have

$$\boldsymbol{x} = \boldsymbol{y} + \frac{\max(0, r_+)}{\boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{\mu}^e} V^{-1}\boldsymbol{\mu}^e$$

(The risk free part x_0 is given by (19c).)

If no additional profit is demanded, one chooses $\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{y}}$.

If $\widetilde{\boldsymbol{y}} = \widetilde{\boldsymbol{0}}$ we have the solution from theorem 2.26.

2.6 Capital Asset Pricing Model

Motivation: Where do the μ_i come from? So far they where given exegenousely, but a market equilibrium should allow for a endogenous determination of μ_i. The model should tell us what the right prices are.

Assume $r_{\text{gmv}} \neq \mu_0$. Then proposition 2.29

implies the existence of the unique tangential portfolio

$$\boldsymbol{x}_{\tan} = \frac{r^e_{\tan}}{\boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{\mu}^e} V^{-1}\boldsymbol{\mu}^e$$

$$\text{where } r^e_{\tan} = \frac{\boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{\mu}^e}{\boldsymbol{\mu}^{e\top} V^{-1}\boldsymbol{e}}$$

$$\text{and } \boldsymbol{x}^\top_{\tan}\boldsymbol{e} = 1$$

Note that $r^e_{\tan} > 0$.

Henceforth every minimum variance portfolio $\widetilde{\boldsymbol{x}}_{r^e}$ is a linear combination of the tangential portfolio $\widetilde{\boldsymbol{x}}_{\tan}$ and the risk free portfolio $\widetilde{\boldsymbol{e}}_0 = (1, 0, \ldots, 0)$.

$$\widetilde{\boldsymbol{x}}_{r^e} = \left(1 - \frac{r^e}{r^e_{\tan}}\right)\widetilde{\boldsymbol{e}}_0 + \frac{r^e}{r^e_{\tan}}\widetilde{\boldsymbol{x}}_{\tan}$$

Assumption 2.31. (Market Model)

Supply Assume $n+1$ financial assets satisfying assumptions 2.25. The total value of asset j at time 0 is given by M_j.

Demand We have N financial agents all holding a minimum variance portfolio $\widetilde{\boldsymbol{x}}^{(i)} \in \mathbb{R}^{n+1}$ and having initial wealth w_i.

Assumption 2.32. (Economic Principle) We have a risk exchange economy with market clearance condition; i.e. supply equals demand.

Definition 2.33. The *market portfolio* $\widetilde{\boldsymbol{x}}^{\mathrm{M}}$ is given by

$$x_j^{\mathrm{M}} = \frac{M_j}{M} \quad \text{where} \quad M = \sum_{j=0}^{n} M_j$$

Each agent holds a minimum variance portfolio.

$$w_i \left(\left(1 - \frac{r_i^e}{r_{\tan}^e} \right), \frac{r_i^e}{r_{\tan}^e} \boldsymbol{x}_{\tan} \right)$$

The total value of risky assets at time 0 is

$$\sum_{i=1}^{N} w_i \frac{r_i^e}{r_{\tan}^e} \boldsymbol{x}_{\tan}$$

The market clearing condition implies

1. $\boldsymbol{x}_{\tan} = \boldsymbol{x}^{\mathrm{M}}$
2. $\displaystyle\sum_{i=1}^{N} w_i \frac{r_i^e}{r_{\tan}^e} = \sum_{j=1}^{n} M_j$

Theorem 2.34. (Capital Asset Pricing Model CAPM) Under assumptions 2.31 and 2.32 we have

$$\mu_j = \mu_0 + \beta_j (r^{\mathrm{M}} - \mu_0)$$

where

$$r^{\mathrm{M}} = \mathrm{E}\Big[R_p(\widetilde{\boldsymbol{x}^{\mathrm{M}}})\Big] \quad \text{and} \quad \beta_j = \frac{\mathrm{Cov}\big[R_j, R_p(\boldsymbol{x}^{\mathrm{M}})\big]}{\mathrm{Var}[R_p(\boldsymbol{x}^{\mathrm{M}})]}$$

Proof. Because of the market clearing condition we can replace the tangent portfolio $\widetilde{\boldsymbol{x}}_{\tan}$ with the market portfolio $\widetilde{\boldsymbol{x}}^{\mathrm{M}}$.

Then the theorem follows straightforward from theorem 2.30. □

Remark. (CAPM)

1. The theorem above does not really tell wether the prices are exogenous or endogenous, but rather gives a balance condition for prices μ_j.
2. Assume the market is large or single assets small, such that a single asset does not really influence the market return r^{M}.

   In practice the following is done:

(a) Determine β_j. (Covariance matrix)

(b) Determine r^{M}. (Expected market return) “growth of the economy”.

(c) Calculate prices assuming independence of the market on single asset prices.

3. CAPM has many unrealistic assumptions:

   - closed market
   - market clearing condition
   - all agents are mean variance optimizers
   - all agents estimate the same parameter values

4. The parameters β_j measure the systemic risk.

3 Arbitrage Pricing Theory (APT)

Remark. In Chapter 1-3 we have two point models two time-points t_0: invest, t_1: observe return. The idea with CAPM is to generalize that and use distributions rather than expected values:

$$R_j = \mu_0 + \underbrace{\beta_j(R_m - \mu_0)}_{\text{systematic risk}} + \underbrace{\varepsilon_j}_{\text{idiosyncratic component for asset } j}$$

where $\mathrm{E}[\varepsilon_j] = 0$. Thus we have:

$$\mathrm{E}[R_j] = \mu_0 + \beta_j(r_\mu - \mu_0)$$

3.1 Exact APT, no ideosyncratic risk

Assumption 3.1 (Model)**.** μ_0 : risk free return,

$R_1, \ldots, R_n$: risky assets fullfilling:

- $R_i = \mu_i + \sum_{k=1}^{K} b_{ik} F_k \qquad \forall i = 1, \ldots, n, b_{ik} \in \mathbb{R}$
- $\mathrm{E}[F_k] = 0$
- $\mathrm{Cov}[\boldsymbol{F}] = \Phi$, positive definite $\in \mathbb{R}^{K \times K}$
- $n > K$

Interpretation: Returns R_i are described by K underlying risk factors $F_1, \ldots, F_k$. In Matrix form we can write that as $\boldsymbol{R} = \boldsymbol{\mu} + B\boldsymbol{F}$ with $B \in \mathbb{R}^{n \times k}$. Since $k < n$:

$$\exists \boldsymbol{x} \in \mathbb{R}^n \backslash \{0\} \text{ orthogonal to all columns of B} \tag{20}$$

Expected return for such a portfolio $\boldsymbol{x}$ is:

$$\begin{aligned}\mathrm{E}[R_p(\widetilde{\boldsymbol{x}})] &= \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} = \mu_0 + \boldsymbol{x}^\top(\boldsymbol{\mu} - \mu_0 \boldsymbol{e}) = \mu_0 + \boldsymbol{x}^\top \boldsymbol{\mu}^e \\ \mathrm{Var}[R_p(\widetilde{\boldsymbol{x}})] &= \boldsymbol{x}^\top \mathrm{Var}[\boldsymbol{R}]\, \boldsymbol{x} = \boldsymbol{x}^\top \mathrm{Var}[(B\boldsymbol{F})]\, \boldsymbol{x} = \underbrace{\boldsymbol{x}^\top B}_{=0} \mathrm{Var}[(\boldsymbol{F})] \underbrace{B^\top \boldsymbol{x}}_{=0} = 0\end{aligned}$$

Definition 3.2 (Economic Priciple)**.** $\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$ is an *arbitrage portfolio* if

$$\begin{aligned}&(i) \boldsymbol{x}^\top \boldsymbol{e} = -x_0 \quad \text{net invest } 0 \\ &(ii)\, \mathrm{E}[R_p(\widetilde{\boldsymbol{x}})] > 0 \\ &(iii)\, \mathrm{Var}[R_p(\widetilde{\boldsymbol{x}})] = 0\end{aligned}$$

From (ii) and (iii) follows that $R_p(\widetilde{\boldsymbol{x}}) > 0$ a.s..

Theorem 3.3. Under the assumption of "no arbitrage" (i.e. $\nexists x$, x arbitrage portfolio) we obtain $\exists \lambda = (\lambda_1, \ldots, \lambda_k)$ risk premium such that

$$\mu_i = \mu_0 + \sum_{k=1}^{K} b_{ik}\lambda_k \quad \forall i = 1, \ldots, n$$

Proof. Choose $\widetilde{\boldsymbol{x}}$ as in (20) and assume $\mathrm{E}[R_p(\widetilde{\boldsymbol{x}})] \neq \mu_0$. Without loss of generality we can assume $r := \mathrm{E}[R_p(\widetilde{\boldsymbol{x}})] > \mu_0$. Invest 1 unit into $\widetilde{\boldsymbol{x}}$ and -1 unit into the risk free asset. Thus we have a net investment of 0 with return $r - \mu_0 > 0$, which is a contradiction to the no arbitrage assumption. Thus

$$\begin{aligned} \mathrm{E}[R_p(\widetilde{\boldsymbol{x}})] &= \mu_0 \\ \mu_0 + \boldsymbol{x}^\top(\boldsymbol{\mu} - \mu_0\boldsymbol{e}) &= \mu_0 \end{aligned}$$

$$\Rightarrow \boldsymbol{x}^\top(\boldsymbol{\mu} - \mu_0\widetilde{\boldsymbol{e}}) = 0$$

This holds true fo all portfolios $\boldsymbol{x}$ which satisfies (20). $\boldsymbol{\mu} - \mu_0\boldsymbol{e}$ is in the span of the columns of B. Thus

$$\exists \boldsymbol{\lambda} : \boldsymbol{\mu} - \mu_0\boldsymbol{e} = B\boldsymbol{\lambda} \tag{21}$$

□

Remark (1)**.** The price of risk λ can explicitly be determined by solving the system (21).

Remark (2)**.** If $k = 1$:

$$\mu_i = \mu_0 + b_i\lambda \qquad (\lambda \text{ is a scalar})$$

Compare this to CAPM:

$$\mu_i = \mu_0 + \beta_i r^e_\mu.$$

The pricing formula coincides.

Remark (3)**.** The two Models should not be identified since we have two compleletely different economic principles.

3.2 Ideosyncractic Risk

Assumption 3.4 (Model Assumption)**.**

- μ_0 : risk free return
- $R_1, R_2, \ldots$ infinite sequence of risky returns.
- $\boldsymbol{R}^N = (R_1, \ldots, R_N) = \boldsymbol{\mu}^N + B^N\boldsymbol{F} + \varepsilon^N$, with:
  - $\boldsymbol{F} = (F_1, \ldots, F_K)$ market risk factors.
  - B^N are the first N rows of an $\infty \times K$- Matrix.

- $\boldsymbol{\varepsilon}^N = (\varepsilon_1, \ldots, \varepsilon_N)$ with $\mathrm{E}[\varepsilon_i] = 0$, $\mathrm{Cov}[\boldsymbol{\varepsilon}] = \Omega^N$ positive definite all eigenvalues bounded uniformly (in N) by $\boldsymbol{\lambda}$
- $\mathrm{Cov}[\varepsilon_i, F_k] = 0$
- $\mathrm{Cov}[\boldsymbol{F}]$ pos definite
- $\mathrm{E}[F_k] = 0$

Remark. Hopefully the main behaviour can be explained by a finite number K of market risk factors such that the remaining (ideosyncratic parts) ε_i are sufficiently small.

Definition 3.5. An *asymptotic arbitrage opportunity* is a sequence of portfolios $\boldsymbol{w}^N \in \mathbb{R}^N$ such that

(i) $\boldsymbol{w}^N \boldsymbol{e}^N = 0$ net investment zero

(ii) $\limsup_{N\to\infty} \mathrm{E}\big[(\boldsymbol{w}^N)^\top R^N\big] \geq \delta > 0$

(iii) $\lim_{N\to\infty} \mathrm{Var}\big[(\boldsymbol{w}^N)^\top R^N\big] = 0$

Theorem 3.6. Under previous model assumptuions and under exclusion of asymptotic arbitrage opportunities. There exists $\widetilde{\boldsymbol{\lambda}}^N = (\lambda_0^N, \ldots, \lambda_K^N)$ of risk factors such that for $N > K$:

$$\mu_i = \lambda_0^N + \sum_{k=1}^{K} b_{ik}\lambda_k^N + v_i^N, \quad \forall i = 1, \ldots, N$$

sucht that the error terms v_i^N satisfy

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (v_i^N)^2 = 0 \tag{22}$$

Idea of the proof is choose in a smart way the sequence $(\widetilde{\boldsymbol{\lambda}}^N)_{N\geq K}$ and show that (22) is satisfied.

Excursion

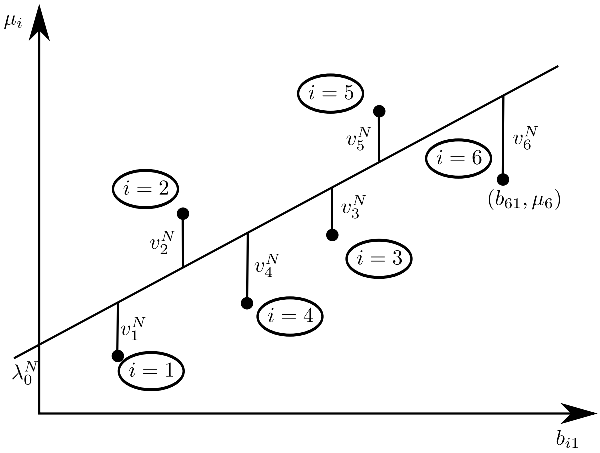

Figure 5: Linear regression

a) Linear regression problem. Assume $K = 1$. $\mu_i = \lambda_0^N + b_{i1}\lambda_1 + v_i^N$.

b) Multivariate linear regression

$$\boldsymbol{v}^N = \boldsymbol{\mu}^N - (\lambda_0^N \boldsymbol{e}^N + B^N \boldsymbol{\lambda}^N)$$

Choose $\tilde{\boldsymbol{\lambda}}^N$ such that $\|v\|_2$ is minimal. Optimal least squares methods (OLS). Solve:

$$(1) \quad \sum_{i=1}^{N} v_i^N = 0 \tag{23}$$

$$(2) \quad \sum_{i=1}^{N} v_i^N b_{ik} = 0 \quad \forall k = 1, \dots, K \tag{24}$$

Choose this $\widetilde{\boldsymbol{\lambda}}^N$ and show that (22) is fullfield. Define $\boldsymbol{w}^N = \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} \boldsymbol{v}^N$ then (23) implies:

$$\begin{aligned}
&(1)(\boldsymbol{w}^N)^\top e^N = 0 \\
&(2)(\boldsymbol{w}^N)^\top \boldsymbol{R}^N = \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} (\boldsymbol{v}^N)^\top (\boldsymbol{\mu}^N + B^N \boldsymbol{F} + \varepsilon^N) \\
&= \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} \left[(\boldsymbol{v}^N)^\top \mu^N + \underbrace{(\boldsymbol{v}^N)^\top B^N \boldsymbol{F}}_{=0} + (\boldsymbol{v})^{N\top} \varepsilon^N \right]
\end{aligned}$$

Calculating the expected value:

$$
\begin{aligned}
\mathrm{E}\Big[(\boldsymbol{w}^N)^\top \boldsymbol{R}^N\Big] &= \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} \left[(\boldsymbol{v}^N)^\top \mu^N + \underbrace{\mathrm{E}\big[(\boldsymbol{v}^N)^\top \varepsilon^N\big]}_{=0} \right] \\
&= \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} (\boldsymbol{v}^N)^\top \left[\boldsymbol{v}^N + \lambda_0^N e^N + B^N \lambda^N\right] \\
&= \frac{1}{\sqrt{N}} \frac{1}{\|\boldsymbol{v}^N\|_2} (\boldsymbol{v}^N)^\top \boldsymbol{v}^N + \underbrace{\lambda_0^N (\boldsymbol{v}^N)^\top e^N}_{=0 \text{ with } (23)(1)} + \underbrace{(\boldsymbol{v}^N)^\top B^N \lambda^N}_{=0 \text{ with } (23)(2)} \\
&= \frac{1}{\sqrt{N}}
\end{aligned}
$$

and the variance:

$$
\begin{aligned}
\mathrm{Var}\Big[(\boldsymbol{w}^N)^\top \boldsymbol{R}^N\Big] &= \frac{1}{N} \frac{1}{\|\boldsymbol{v}^N\|_2^2} \mathrm{Var}\big[(\boldsymbol{w}^N)^\top \varepsilon^N\big] \\
&= \frac{1}{N} \frac{1}{\|\boldsymbol{v}^N\|_2^2} (\boldsymbol{w}^N)^\top \underbrace{\mathrm{Cov}\big[\boldsymbol{\varepsilon}^N\big]}_{\Omega^N} v^N \\
&\overset{\leq}{\text{eigenvalues unif. bounded}} \frac{1}{N} \frac{1}{\|\boldsymbol{v}^N\|_2^2} \bar{\lambda} \|\boldsymbol{v}^N\|_2^2 = \frac{\bar{\lambda}}{N} \to 0.
\end{aligned}
$$

the $\bar{\lambda}$ is uniformly bounded.

Proof. Assume (22) does not hold true for our choice $\widetilde{\boldsymbol{\lambda}}^N$ and v^N. Then (i) and (ii) are fullfield and there exists $\delta > 0$ with

$$
\limsup \mathrm{E}\Big[(\boldsymbol{w}^N)^\top \boldsymbol{R}^N\Big] \geq \delta
$$

which contradicts the no arbitrage assumption. □

4 Multiperiod Models

We consider a discrete Time Model with finite time horizon, with $t \in \{0, 1, \dots, T\}$, $T < \infty$ and a cashflow $\boldsymbol{c} = (c_1, \dots, c_T)$.

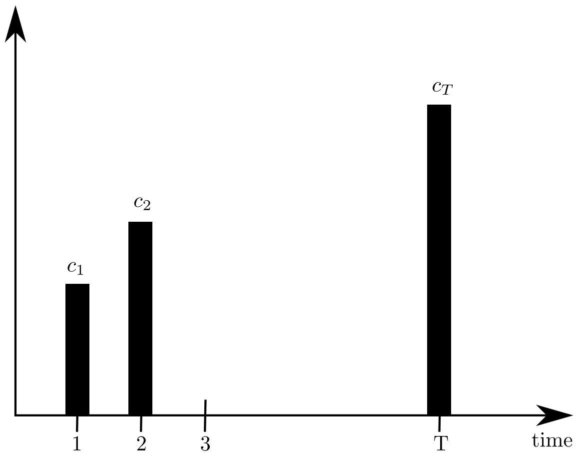

Figure 6: Cashflow with finite time horizon.

The aim is to find a price we are willing to pay at time 0 for this cashflow.

4.1 Deterministic Cashflows and Arbitrage

Definition 4.1. $\boldsymbol{v} \in \mathbb{R}^N$

- $\boldsymbol{v} \geq 0$ iff $v_i \geq 0, \forall i = 1, \dots, N$, iff $\boldsymbol{v} \in \mathbb{R}^N_+ \cup \{0\}$
- $\boldsymbol{v} > 0$ iff $\boldsymbol{v} \geq 0$ and $\exists i : v_i > 0$, iff $\boldsymbol{v} \in \mathbb{R}^N_+$
- $\boldsymbol{v} \gg 0$ iff $v_i > 0$ $\forall i = 1, \dots, N$, iff $\boldsymbol{v} \in \mathbb{R}^N_{++}$

Definition 4.2 (Security Market)**.** A security market is a pair $(\boldsymbol{\Pi}, C)$ with $\boldsymbol{\Pi} \in \mathbb{R}^N$ and $C \in \mathbb{R}^{N \times T}$.

$$\boldsymbol{\Pi} = \begin{bmatrix} \Pi_1 \\ \vdots \\ \Pi_i \\ \vdots \\ \Pi_n \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & \dots & c_{1T} \\ \vdots & & \vdots \\ c_{i1} & \dots & c_{iT} \\ \vdots & & \vdots \\ c_{N1} & \dots & c_{NT} \end{bmatrix}$$

Each row in C represents a cashflow. We thus have N cashflows with corresponding prices $\boldsymbol{\Pi}$ at time zero.

Definition 4.3. A prortfolio strategy is a vector $\boldsymbol{\theta} \in \mathbb{R}^N$. The cashflow generated by such a portfolio strategy $\boldsymbol{\theta}$ is:

$$C^\top \boldsymbol{\theta} = (\sum_{i=1}^{N} \theta_i c_{i1}, \dots, \sum_{i=1}^{N} \theta_i c_{iT})^\top.$$

The price of $C^\top \boldsymbol{\theta}$ at time 0 is $\boldsymbol{\Pi}^\top \boldsymbol{\theta} = \sum_i^N \theta_i \Pi_i$

Definition 4.4 (Arbitrage opportunity)**.** $\boldsymbol{\theta} \in \mathbb{R}^N$ is an arbitrage opportunity if it satisties one of the two conditions:

1. $\boldsymbol{\Pi}^\top \boldsymbol{\theta} = 0$ and $C^\top \boldsymbol{\theta} > 0$ or
2. $\boldsymbol{\Pi}^\top \boldsymbol{\theta} < 0$ and $C^\top \boldsymbol{\theta} \geq 0$

Definition 4.5 (Arbitrage-free)**.** A security market $(\boldsymbol{\Pi}, C)$ is arbitrage-free if it contains no arbitrage opportunity $\boldsymbol{\theta}$.

Lemma 4.6. Let A be an $m \times n$ Matrix then exactly one of the follwoing statements holds

1. $\exists \boldsymbol{x} \in \mathbb{R}^m_{++}$ such that $A\boldsymbol{x} = 0$
2. $\exists \boldsymbol{y} \in \mathbb{R}^n$ such that $y^\top A > 0$

Proof. Stiemke Lemma. See elsewhere. □

Remark. Another proof would be given by the seperating hyperplane theorem. Farkas?

Theorem 4.7. The security market $(\boldsymbol{\Pi}, C)$ is arbitrage free if and only if

$$\exists \boldsymbol{d} \in \mathbb{R}^T_{++} \text{ with } \boldsymbol{\Pi} = C\boldsymbol{d}.$$

Interpretation: $\boldsymbol{d}$ is the vector of discount factors. A cashflow of 1 at time i has the price d_i.

Proof. Define

$$A = \begin{bmatrix} -\Pi_1 & c_{11} & \dots & c_{1T} \\ \vdots & \vdots & & \vdots \\ -\Pi_n & c_{n1} & \dots & c_{nT} \end{bmatrix}$$

Apply Stiemkes Lemma:

$$
\begin{aligned}
\text{from i)} \exists \boldsymbol{x} \in \mathbb{R}_{++}^{T+1} &: A\boldsymbol{x} = 0 \\
\Leftrightarrow \exists \boldsymbol{x} \in \mathbb{R}_{++}^{T+1} &: -\Pi_i x_0 = \sum_{t=1}^{T} x_t c_{it} = 0 \qquad \forall i = 1, \dots, n \\
\Leftrightarrow \exists \boldsymbol{x} \in \mathbb{R}_{++}^{T+1} &: \Pi_i = \sum_{t=1}^{T} \frac{x_t}{x_0} c_{it} \qquad \forall i = 1, \dots, n
\end{aligned}
$$

define $d_t = \frac{x_t}{x_0}$

$$
\Leftrightarrow \exists \boldsymbol{d} \in \mathbb{R}_{++}^{T+1} : \Pi_i = \sum_{t=1}^{T} d_t c_{it} \qquad \forall i = 1, \dots, n
$$

It remains to prove that no-arbitrage assumption rules out being in case (ii) of Stiemkes Lemma. Assume $\exists \boldsymbol{y} \in \mathbb{R}^N$ such that $y^\top A > 0$. There are now to cases:

1. either $(\boldsymbol{y}^\top A)_1 > 0$ and $(\boldsymbol{y}^\top A)_k \geq 0$ for $k = 2, \dots, T+1 \Leftrightarrow$ arbitrage definiton case (2).
2. or $(\boldsymbol{y}^\top A)_1 = 0$ and $(\boldsymbol{y}^\top A)_k \geq 0$ for $k = 2, \dots, T+1$ and $\exists l : (\boldsymbol{y}^\top A)_l > 0$ $\Leftrightarrow$ arbitrage definiton case (1).

We have shown that in Stemkes Lemma (i) $\Leftrightarrow$ there exists a discount factor. (ii) $\Leftrightarrow$ there exists an arbitrage opprotunity. □

Remark. In the finite dimensional setup "No arbitrage" $\Leftrightarrow$ "There are discount factors". Probabilities play no role.

Remark. In more general models: discrete time but infinite probability space or in continous time models we can usually show:

$$\exists \text{martingale measuere} \Rightarrow \text{no arbitrage}$$

For the case $\Leftarrow$ it is important which definition of arbitrag implies the existence of a martingale measure. (See: [DS93]. Why are infinite probabilities space so much harder? For example in continous time one needs to exclude "doubling strategies", which plays infinitely many lotteries each doubling the investment until the first win.

Remark. We do not say anything about the uniquenss of $\boldsymbol{d}$.

Definition 4.8. The market $(\boldsymbol{\Pi}, C)$ is called complete if for any $\boldsymbol{y} \in \mathbb{R}^T$ there exists $\boldsymbol{\theta} \in \mathbb{R}^N$ such that $C^\top \boldsymbol{\theta} = \boldsymbol{y}$. (i.e. if we can replicate any cashflow $\boldsymbol{y}$)

Proposition 4.9. The market is complete if an only if there exists an unique $\boldsymbol{d} \in \mathbb{R}_{++}^T$ such that $\boldsymbol{\Pi} = C\boldsymbol{d}$.

Proof. No proof given. □

Remark. In that case C can be choosen as a regular $T \times T$-matrix.

4.2 Term Structures in Discrete Time

Assume $(\mathbf{\Pi}, C)$ is arbitrage free and complete. It follows that there exist unique discount factors $\boldsymbol{d}$.

Definition 4.10. A zero-coupon bond (ZCB) with maturity $t \in \{1, \dots, T\}$ is given by $e_t = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^T$, where the 1 is at the t-th position.

Proposition 4.11. The price of ZCB(t) at time 0 is given by d_t.

Proof. Due to completeness $\exists \boldsymbol{\theta}_t : e_t = C^\top \boldsymbol{\theta}_t$ It follows that:

$$\text{Price}(\text{ZCB}(t)) = \Pi^\top \boldsymbol{\theta}_t = (C\boldsymbol{d})^\top \boldsymbol{\theta}_t = \boldsymbol{d}^\top C^\top \boldsymbol{\theta}_t = \boldsymbol{d}^\top \boldsymbol{e}_t = d_t$$

□

Corollary 4.12. The price of any casfhlow $\boldsymbol{c} \in \mathbb{R}^T$ is given by

$$\text{Price}_0(\boldsymbol{c}) = \sum_{t=1}^{T} c_t d_t$$

Proof. Write $\boldsymbol{c} = \sum c_t e_t$ and use linearity. □

All we need in this section is arbitrage-free and complete. We have not assumed any monotonicity properties on $\boldsymbol{d}$. You would guess $d_{t+1} < d_t$, but this is not assumed or implied here. We use the following terminology to normalize $d_0 = 1$.

Definition 4.13. Assume $\boldsymbol{d} \in \mathbb{R}^T_{++}$ is given as above. The spot rate at time 0 is $r_0 = \frac{1}{d_1} - 1$. The (one period) t-forward rate at time zero is $f(0,t) = \frac{d_t}{d_{t+1}} - 1$.

Definition 4.14. Yield-to-maturity of ZCB(t):

$$y(0,t) = \left(\frac{1}{d_t}^{1/t}\right) - 1 \Leftrightarrow d_t = (1 + y(0,t))^{-t}$$

Term structure at time 0 is given by $(y(0,1), y(0,2), \dots, y(0,T))$.

4.3 Duration

$\boldsymbol{c} \in \mathbb{R}^T$: cashflow, $\boldsymbol{d} \in \mathbb{R}^T_{++}$ given.

$$\text{Price}_0(\boldsymbol{c}) = \sum_{t=1}^{T} c_t d_t = \sum_{t=1}^{T} \frac{c_t}{(1 + y(0,t))^t}$$

In old models one assumes often $y(0,t) = r$

$$\text{Price}_0(\boldsymbol{c}) = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t} =: \Pi_0(\boldsymbol{c}, r)$$

$$\frac{d\Pi_0(\boldsymbol{c},r)}{dr} = -\sum t\frac{c_t}{(1+r)^{t+1}} = -\frac{1}{1+r}\sum_{t=1}^{T} t\frac{c_t}{(1+r)^t}$$

Relative change:

$$\frac{\frac{d\Pi_0(\boldsymbol{c},r)}{dr}}{\Pi_0(\boldsymbol{c},r)} = -\frac{1}{1+r} \underbrace{\frac{\sum t\frac{c_t}{(1+r)^t}}{\sum \frac{c_t}{(1+r)^t}}}_{=:D(\boldsymbol{c},r)\text{Maucaly Duration}}$$

"Average time of the payouts c_t" or "Expected value of the payout time" $e_D = \mathrm{E}[c_t]$ for a certain probability measure. We can do a Taylor expansion on:

$$\Pi_0(\boldsymbol{c}, r+\Delta r) = \Pi_0(\boldsymbol{c},r) - \Pi_0(\boldsymbol{c},r)\frac{D(\boldsymbol{c},r)}{1+r}\Delta r + o(\Delta r)$$

Assume $\boldsymbol{c}$ are liabilities and $\boldsymbol{y}$ are cashflow of our assets. How should we choose $\boldsymbol{y}$? For ALM buy $\boldsymbol{y}$ such that

$$\begin{aligned}\Pi_0(\boldsymbol{y},r) =& \Pi_0(\boldsymbol{c},r)\\ D(\boldsymbol{c},r) =& D(\boldsymbol{y},r)\end{aligned}$$

Remark. Similar approximations are available for non-constant term structure curves and interest rate shocks. See (Shiu 1987, Fong-Vasicek 1984).

4.4 Stochastic Cashflow in Discrete Time

Assumption 4.15. Assume a final time horizon $T \in \mathbb{N}$, $(\Omega, \mathcal{F}, P, \mathbb{F})$ a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,T}$ where $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$, $\mathcal{F}_t$ a σ-field.

Assume $\boldsymbol{X} = (X_0, \ldots, X_T)$ is a $\mathbb{F}$-adapted cashflow on $(\Omega, \mathcal{F}, P)$

The goal will be to calculate the price process for any such cashflow $\boldsymbol{X}$ under an appropriate valuation functional.

Assumption 4.16. (General Assumptions) Assume all components X_t of $\boldsymbol{X}$ are square-integrable i.e. $\mathrm{E}\left[X_t^2\right] < \infty$ for all $t \in \{0, \ldots, T\}$.

Remark. $L^2_{T+1}(\Omega, \mathcal{F}, P)$ is a hilbert space:

- $\mathrm{E}\left[X_t^2\right] < \infty$ for all t, for all $\boldsymbol{X} \in L^2_{T+1}$
- $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \mathrm{E}\left[\boldsymbol{X}^\top \boldsymbol{Y}\right]$ scalar product
- $\|\boldsymbol{X}\| = \sqrt{\langle \boldsymbol{X}, \boldsymbol{X} \rangle}$ norm

If $\|\boldsymbol{X} - \boldsymbol{Y}\| = 0$ then $\boldsymbol{X} = \boldsymbol{Y}$ P-almost surely. In that case we identify $\boldsymbol{X}$ with $\boldsymbol{Y}$.

Definition 4.17. (Notation)

- $\boldsymbol{X} \geq 0$ if and only if $X_k \geq 0$ P-almost surely, for all k.
- $\boldsymbol{X} > 0$ if and only if $\boldsymbol{X} \geq 0$ and there is a k such that $P(X_k > 0) > 0$.
- $\boldsymbol{X} \gg 0$ if and only if for all k: $X_k > 0$ P-almost surely.

4.4.1 Market Consistent Valuation at Time Zero

Assumption 4.18. Assume a functional $Q_0 : L^2_{T+1}(\Omega, \mathcal{F}, P) \to \mathbb{R}$ with the properties

1. (Positivity) $\boldsymbol{X} > 0 \Rightarrow Q_0(\boldsymbol{X}) > 0$.
2. (Continuity) If $(\boldsymbol{X}^{(i)})_{i=1,2,\ldots}$ is a convergent sequence in L^2_{T+1}:
$$\boldsymbol{X}^{(i)} \underset{L^2}{\overset{i\to\infty}{\longrightarrow}} \boldsymbol{X} \quad \Longrightarrow \quad Q_0(\boldsymbol{X}^{(i)}) \overset{i\to\infty}{\longrightarrow} Q_0(\boldsymbol{X}).$$
3. (Linearity) $Q_0(a\boldsymbol{X} + b\boldsymbol{Y}) = aQ_0(\boldsymbol{X}) + bQ_0(\boldsymbol{Y})$.

Definition 4.19. If Q_0 satisfies the assumptions above we call it a *pricing functional* and we call $Q_0(\boldsymbol{X})$ the *price of* $\boldsymbol{X}$ *at time* 0.

Lemma 4.20. In the assumptions above positivity and linearity imply continuity.

Proof. See Wüthrich–Bühlmann–Furrer (2008) Springer □

Theorem 4.21. (Riesz Representation Theorem) If Q_0 is a pricing functional on $L^2_{T+1}(\Omega, \mathcal{F}, P)$ then there exists $\boldsymbol{\varphi} \in L^2_{T+1}(\Omega, \mathcal{F}, P)$ such that for all $\boldsymbol{X} \in L^2_{T+1}(\Omega, \mathcal{F}, P)$: $Q_0(\boldsymbol{X}) = \langle \boldsymbol{\varphi}, \boldsymbol{X} \rangle$.

Definition 4.22. The $\boldsymbol{\varphi}$ in the theorem above is called *deflator*.

In economic literature it is often called *state price density* and in financial mathematics *pricing kernel*.

Remark.

- Positivity of Q_0 implies $\boldsymbol{\varphi} \gg 0$.
- On the subspace of all $\mathbb{F}$-adapted cashflows $\boldsymbol{\varphi}$ can be chosen $\mathbb{F}$-adapted.
  Indeed: define $\tilde{\varphi}_k = \mathrm{E}[\varphi_t | \mathcal{F}_t]$. $\widetilde{\boldsymbol{\varphi}}$ is a deflator and it is $\mathbb{F}$-adapted.
  If $\boldsymbol{X}$ is $\mathbb{F}$-adapted then $\langle \boldsymbol{X}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{X}, \widetilde{\boldsymbol{\varphi}} \rangle$.
- The $\mathbb{F}$-adapted deflator is unique.
  Indeed: assume two deflators $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^*$
  $$\langle \boldsymbol{\varphi}, \boldsymbol{X} \rangle = \langle \boldsymbol{\varphi}^{\boldsymbol{*}}, \boldsymbol{X} \rangle \text{ for all } \boldsymbol{X}$$
  In particular for $\boldsymbol{X} = \boldsymbol{\varphi} - \boldsymbol{\varphi}^*$.
  $$0 = \langle \boldsymbol{\varphi} - \boldsymbol{\varphi}^*, \boldsymbol{\varphi} - \boldsymbol{\varphi}^* \rangle = \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^*\|^2$$
  And thus $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$
- In the following we will always use the $\mathbb{F}$-adapted deflator $\boldsymbol{\varphi}$ to represent Q_0.

4.4.2 Understanding Deflators

1. In theorem 4.7 we have seen security market pricing iff there exists $\boldsymbol{d} \in \mathbb{R}^T_{++}$ such that $\Pi = C\boldsymbol{d}$. (finite space model)
   In theorem 4.21: Q_0 pricing function iff there exists $\boldsymbol{\varphi} \gg 0$.
2. Attention: In general $\boldsymbol{\varphi}$ and $\boldsymbol{X}$ cannot be decoupled.
   $$\sum_{t=0}^{T} \mathrm{E}[\varphi_t X_t] \neq \sum_{t=0}^{T} \mathrm{E}[\varphi_t] \, \mathrm{E}[X_t]$$
3. The price of a ZCB with maturity t is
   $$Q_0(\boldsymbol{e}_t) = \langle \boldsymbol{\varphi}, \boldsymbol{e}_t \rangle = \mathrm{E}[\varphi_t] = P(0, t).$$
   It is $\mathcal{F}_0$-measurable, i.e. known at time zero, (deterministic discounting) whereas φ_t is $\mathcal{F}_t$ measurable and allows for stochastic discounting.

4.4.3 Pricing at Positive Time

Definition 4.23. For $\boldsymbol{X} \in L^2_{T+1}(\Omega, \mathcal{F}, P, \mathbb{F})$ we define its *price at time t* by

$$Q_t(\boldsymbol{X}) = \frac{1}{\varphi_t} \operatorname{E}\left[\sum_{s=0}^{T} \varphi_s X_s \,\middle|\, \mathcal{F}_t \right].$$

Observe that $Q_t((X))$ is $\mathcal{F}_t$-measurable.

Lemma 4.24. $(\varphi_t Q_t(\boldsymbol{X}))_t$ is an $(\Omega, \mathcal{F}, P, \mathbb{F})$-martingale.

Proof.

$$\begin{aligned}
\operatorname{E}[\varphi_{t+1} Q_{t+1}(\boldsymbol{X}) | \mathcal{F}_t] &= \operatorname{E}\left[\varphi_{t+1} \frac{1}{\varphi_{t+1}} \operatorname{E}\left[\sum_{s=0}^{T} \varphi_s X_s \,\middle|\, \mathcal{F}_{t+1} \right] \,\middle|\, \mathcal{F}_t \right] \\
&= \operatorname{E}\left[\operatorname{E}\left[\sum_{s=0}^{T} \varphi_s X_s \,\middle|\, \mathcal{F}_{t+1} \right] \,\middle|\, \mathcal{F}_t \right] \\
&= \operatorname{E}\left[\sum_{s=0}^{T} \varphi_s X_s \,\middle|\, \mathcal{F}_t \right] \\
&= \varphi_t Q_t(\boldsymbol{X})
\end{aligned}$$

□

Theorem 4.25. The pricing system $(\varphi_t Q_t(\boldsymbol{X}))_t$ on $L^2(\Omega, \mathcal{F}, P, \mathbb{F})$ is arbitrage-free for an appropriate definition of arbitrage.

Remark. A rigorous version of this theorem can be found in [DS98] and in M. Schweizers lecture on financial mathematics.

Example.

$$P(s,t) = \frac{1}{\varphi_s} \operatorname{E}[\varphi_t | \mathcal{F}_s] = \operatorname{E}\left[\frac{\varphi_t}{\varphi_s} \,\middle|\, \mathcal{F}_s \right] \qquad (s \leq t)$$

4.4.4 Equivalent Martingale Measure

In financial math one typically chooses an appropriate unit (called numéraire) such that formulae become simple. Here: bank-account numéraire.

Definition 4.26. The *spot rate process* $(R(t))_t$ is given by

$$R(t) = -\log P(t, t+1) = -\log \left(\frac{1}{\varphi_{t+1}} \operatorname{E}[\varphi_{t+1} | \mathcal{F}_t] \right).$$

Remark. The spot rate $R(t)$ is $\mathcal{F}_t$-measurable.

If the time step is one year then $R(t)$ corresponds to the one year risk free asset.

The spot rate process plays the role of the bank account: invest 1 at time zero, this then provides the value B_t at time t.

$$B_t = \exp R(0) \cdot \exp R(1) \cdot \ldots \cdot R(t-1) = \exp \sum_{s=0}^{t-1} R(s)$$

The goal of this section is to study the price process relative to the *bank account numéraire* $\frac{1}{B_t}$

We have seen that $(\varphi_t Q_t(\boldsymbol{X}))_t$ is an $(\Omega, \mathcal{F}, P, \mathbb{F})$-martingale, now what about $(B_t^{-1} Q_t(\boldsymbol{X}))_t$?

Note that φ_t is $\mathcal{F}_t$-measurable, observable at the end of period $[t-1, t]$, whereas B_t is $\mathcal{F}_{t-1}$-measurable, observable at $t-1$. This property is called *previsible.*

Definition 4.27. $\xi_t = \varphi_t B_t$ for $t = 1, \ldots, T$ and $\xi_0 = 0$.

Lemma 4.28. $\boldsymbol{\xi} \gg 0$ and $(\xi_t)_t$ is an $(\Omega, \mathcal{F}, P, \mathbb{F})$-martingale and $\mathrm{E}[\xi_T] = 1$.

Proof. Positivity: $\boldsymbol{\varphi} \gg 0$ and $\boldsymbol{B} \gg 0$ therefore also $\boldsymbol{\xi} \gg 0$.

Martingale property: observe that

$$\xi_t = \varphi_t B_t = \varphi_{t-1} B_{t-1} \frac{\varphi_t}{\varphi_{t-1}} \exp R(t-1)$$

$$\begin{aligned}
\mathrm{E}[\xi_t | \mathcal{F}_{t-1}] &= \varphi_{t-1} B_{t-1} e^{R(t-1)} \frac{1}{\varphi_{t-1}} \mathrm{E}[\varphi_t | \mathcal{F}_{t-1}] \\
&= \varphi_{t-1} B_{t-1} e^{R(t-1)} P(t, t+1) \\
&= \varphi_{t-1} B_{t-1}
\end{aligned}$$

Expected Value:

$$\mathrm{E}[\xi_T] = \mathrm{E}[\xi_1] = B_1 \, \mathrm{E}[\varphi_1] = 1$$

□

Note that ξ_t is a density process with respect to P. ξ_t is a density.

Definition 4.29. The equivalent probability measure $\mathbb{P}_0$ is given by the Radon–Nikodym derivate

$$\frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}P} = \xi_T = \varphi_T B_T.$$

In other words

$$\mathbb{P}_0(A) = \int_A \xi_T \mathrm{d}P \quad \text{for all } A \in \mathcal{F}$$

Corollary 4.30.

$$\mathrm{E}_P\left[\frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}P} | \mathcal{F}_t\right] = \mathrm{E}_P[\xi_T | \mathcal{F}_t] = \xi_t$$

Lemma 4.31. For $s < t$ the following holds P-almost surely:

$$\mathrm{E}_{\mathbb{P}_0}[Q_t(\boldsymbol{X})|\mathcal{F}_s] = \frac{1}{\xi_s}\,\mathrm{E}_P[\xi_t Q_t(\boldsymbol{x})|\mathcal{F}_s]$$

Proof. For all $A \in \mathcal{F}_s$ the following holds:

$$\begin{aligned}\mathbb{P}_0(A) &= \mathrm{E}_P[\xi_T \mathbf{1}_A] \\ &= \mathrm{E}_P[\mathrm{E}_P[\xi_T \mathbf{1}_A|\mathcal{F}_s]] \\ &= \mathrm{E}_P[\mathbf{1}_A\,\mathrm{E}_P[\xi_T|\mathcal{F}_s]] \\ &= \mathrm{E}_P[\mathbf{1}_A \xi_s]\end{aligned}$$

$$\begin{aligned}\mathrm{E}_{\mathbb{P}_0}[\mathbf{1}_A Q_t(\boldsymbol{X})] &= \mathrm{E}_P[\xi_T \mathbf{1}_A Q_t(\boldsymbol{X})] \\ &= \mathrm{E}_P[\mathrm{E}_P[\xi_T \mathbf{1}_A Q_t(\boldsymbol{X})|\mathcal{F}_s]] \\ &= \mathrm{E}_P\left[\mathbf{1}_A \xi_s \frac{1}{\xi_s}\,\mathrm{E}_P[\xi_T Q_t(\boldsymbol{X})|\mathcal{F}_s]\right]\end{aligned}$$

And therefore P-almost surely

$$\frac{1}{\xi_s}\,\mathrm{E}_P[\xi_t Q_t(\boldsymbol{X})|\mathcal{F}_s] = \mathrm{E}_{\mathbb{P}_0}[Q_t(\boldsymbol{X})|\mathcal{F}_s]$$

□

Corollary 4.32. The process $(B_t^{-1} Q_t(\boldsymbol{X}))_t$ is an $(\Omega, \mathcal{F}, \mathbb{P}_0, \mathbb{F})$-maringale. (Note the changed probability measure.)

Proof. Applying lemma 4.31 with $s \leftarrow t$, $t \leftarrow t+1$

$$\begin{aligned}\mathrm{E}_{\mathbb{P}_0}[Q_{t+1}(\boldsymbol{X})|\mathcal{F}_t] &= \frac{1}{\xi_t}\,\mathrm{E}_P[\xi_{t+1} Q_{t+1}(\boldsymbol{X})] \\ &= \mathrm{E}_P\left[\frac{\varphi_{t+1}}{\varphi_t} e^{R(t)} Q_{t+1}(\boldsymbol{X})\,\middle|\,\mathcal{F}_t\right] \\ &= e^{R(t)}\frac{1}{\varphi_t}\,\mathrm{E}_P[\varphi_{t+1} Q_{t+1}(\boldsymbol{X})|\mathcal{F}_t]\end{aligned}$$

using the fact that $\varphi_t Q_t(\boldsymbol{X})$ is a P-martingale

$$\begin{aligned}&= e^{R(t)}\frac{1}{\varphi_t}\varphi_t Q_t(\boldsymbol{X}) \\ &= e^{R(t)} Q_t(\boldsymbol{X})\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{E}_{\mathbb{P}_0}\big[B_{t+1}^{-1} Q_{t+1}(\boldsymbol{X})|\mathcal{F}_t\big] &= B_{t+1}^{-1}\,\mathrm{E}_{\mathbb{P}_0}[Q_{t+1}(\boldsymbol{X})|\mathcal{F}_t] \\ &= B_{t+1}^{-1} e^{R(t)} Q_t(\boldsymbol{X}) \\ &= B_t^{-1} Q_t(\boldsymbol{X})\end{aligned}$$

□

In the real-world probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ the process $(\varphi_t Q_t(\boldsymbol{X}))_t$ is a maringale.

In the risk neutral measure or equivalent martingale measure $(\Omega, \mathcal{F}, \mathbb{P}_0, \mathbb{F})$ the process $(B_t^{-1} Q_t(\boldsymbol{X}))_t$ is a martingale.

Remark. Usually pricing is simpler with the bank account numéraire B_t^{-1} and $\mathbb{P}_0$, whereas modelling insurance products is simpler with the deflator φ_t and P.

Corollary 4.33. Let $s < t$. Then

$$P(s,t) = \frac{1}{\varphi_s} \operatorname{E}_P[\varphi_t | \mathcal{F}_s] = \operatorname{E}_{\mathbb{P}_0}\left[\exp\left(-\sum_{u=s}^{t-1} R(u) \right) \middle| \mathcal{F}_s \right]$$

Proof.

$$\begin{aligned}
\operatorname{E}_{\mathbb{P}_0}\left[\exp\left(-\sum_{u=s}^{t-1} R(u) \right) \middle| \mathcal{F}_s \right] &= \operatorname{E}_{\mathbb{P}_0}\left[\frac{B_s}{B_t} \middle| \mathcal{F}_s \right] \\
&= B_s \operatorname{E}_{\mathbb{P}_0}\big[B_t^{-1} P(t,t) | \mathcal{F}_s \big]
\end{aligned}$$

Using the fact that $(B_\tau^{-1} P(\tau, t))_\tau$ is a $\mathbb{P}_0$-martingale

$$\begin{aligned}
&= B_s (B_s^{-1} P(s,t)) \\
&= P(s,t)
\end{aligned}$$

□

4.4.5 Market Price of Risk

The market price of risk will explain the difference between P and $\mathbb{P}_0$.

Assumption 4.34. Let $(\Omega, \mathcal{F}, P, \mathbb{F})$ a filtered probability space. Assume $(\varepsilon_t)_{t=0\ldots T}$ is $\mathbb{F}$-adapted and $\varepsilon_t | \mathcal{F}_{t-1} \sim \mathcal{N}(0,1)$. Assume the short rate dynamics $R(t)$ is given by

$$R(0) = r_0 > 0, R(t) = f(t, R(t-1)) + \varepsilon_t g(t, R(t-1)) \text{ for } t = 1 \ldots T \tag{25}$$

where f and g are sufficiently nice functions.

Our goal will be to find appropriate deflator models and an appropriate equivalent measure $\mathbb{P}_0 \sim P$.

Assumption 4.35. Assume we have a second $\mathbb{F}$-adapted process $(\delta_t)_{t=0\ldots T}$ such that $\delta_t | \mathcal{F}_{t-1} \sim \mathcal{N}(0,1)$ and $\operatorname{Cov}[P]\, \delta_t, \varepsilon_t = \rho$.

Define the deflator

$$\varphi_t = \exp\left(-\sum_{s=1}^{t} \big(R(s-1) + \tfrac{1}{2}\lambda^2(s, R(s-1)) \big) + \sum_{s=1}^{t} \lambda(s, R(s-1)) \delta_s \right) \tag{26}$$

where λ is a sufficiently nice function.

Definition 4.36. The function λ is called *market price of risk.*

Remark. Observe that λ describes the difference between B_t^{-1} and φ_t in the sense that $B_t^{-1} = \varphi_t$ when $\lambda \equiv 0$.

λ describes the aggregate market risk aversion. It is often given exogeneously.

Proposition 4.37. The equations (25) and (26) give a meaningful model i.e.

- φ_t is $\mathbb{F}$-adapted.
- $\varphi_t \in L^2(\Omega, \mathcal{F}, P)$.
- $R(t) = -\log\left(\frac{1}{\varphi_t} \mathrm{E}_P[\varphi_{t+1}|\mathcal{F}_t]\right)$

Proof.

$$\begin{aligned}
&-\log \frac{1}{\varphi_t} \mathrm{E}_P[\varphi_{t+1}|\mathcal{F}_t] \\
&= -\log \mathrm{E}_P\big[\exp(-R(t) - \tfrac{1}{2}\lambda^2(t+1, R(t)\big] + \lambda(t+1, R(t))\delta_{t+1})|\mathcal{F}_t) \\
&= -\log\big(\exp(-R(t) - \tfrac{1}{2}\lambda^2(t+1, R(t)))\, \mathrm{E}_P[\exp(\lambda(t+1, R(t))\delta_{t+1})|\mathcal{F}_t]\big)
\end{aligned}$$

using expected value of a lognormal distributed random variable

$$\begin{aligned}
&= -\log\big(\exp(-R(t) - \tfrac{1}{2}\lambda^2(t+1, R(t)))\exp(0 + \tfrac{1}{2}\lambda^2(t+1, R(t)))\big) \\
&= R(t)
\end{aligned}$$

□

The equivalent martingale measure under previous assumptions is obtained by the Girsanov transformation in discrete time.

$$\frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}P} = \varphi_T B_T = \exp\left(-\frac{1}{2}\sum_s \lambda^2(s, R(s-1)) + \sum_s \lambda(s, R(s-1))\delta_s\right)$$

Lemma 4.38. $\varepsilon_t^* = \varepsilon_t - \lambda\rho$ has, given $\mathcal{F}_{t-1}$ a standard normal distribution under $\mathbb{P}_0$.

Proof. We will proove the lemma by comparing the moment generating function of ε_t^* with the moment generating function of a standard normal distributed random variable. Choose $s \in \mathbb{R}$. The moment generating function of ε_t^* under $\mathbb{P}_0$, given $\mathcal{F}_{t-1}$, is

$$\begin{aligned}
(m_{\mathbb{P}_0}\varepsilon_t^*)(s) &= \mathrm{E}_{\mathbb{P}_0}[\exp \varepsilon_t^* s|\mathcal{F}_{t-1}] \\
&= \exp(-s\lambda\rho)\, \mathrm{E}_{\mathbb{P}_0}[\exp(-s\varepsilon_t)|\mathcal{F}_{t-1}] \\
&= \exp(-s\lambda\rho)\, \mathrm{E}_P\big[\exp(-\tfrac{1}{2}\lambda^2 + \lambda\delta_t + s\varepsilon_t)|\mathcal{F}_{t-1}\big]
\end{aligned}$$

$\lambda = \lambda(t, R(t-1))$ is constant with respect to $\mathcal{F}_{t-1}$.

$$= \exp(-s\lambda\rho - \tfrac{1}{2}\lambda^2)\, \mathrm{E}_P[\exp(\lambda\delta_t + s\varepsilon_t)|\mathcal{F}_{t-1}]$$

using lognormal expectancy and $\lambda\delta_t + s\varepsilon_t|\mathcal{F}_{t-1} \sim \mathcal{N}(0, \lambda^2 + 2\lambda s\rho + s^2)$

$$\begin{aligned} &= \exp(-s\lambda\rho - \tfrac{1}{2}\lambda^2)\exp(\tfrac{1}{2}(\lambda^2 + 2\lambda s\rho + s^2)) \\ &= \exp(\tfrac{1}{2}s) \end{aligned}$$

which is indeed the moment generating function of a standard normal distributed random variable. □

Remark.

$$P(s,t) = \frac{1}{\varphi_s}\,\mathrm{E}_P[\varphi_t|\mathcal{F}_s] = \mathrm{E}_{\mathbb{P}_0}\left[-\sum_{u=s}^{t-1} R(u) \middle| \mathcal{F}_s \right]$$

indeed, from (25) we obtain

$$R(t) = f(t, R(t-1)) + \delta\lambda(t, R(t+1))g(t, R(t-1)) + \varepsilon_t^* g(t, R(t-1))$$

for the short rate dynamics under $\mathbb{P}_0$.

4.4.6 Vasicek Model in Discrete Time

([Vasicek, 1997] in continuous time)

Assumption 4.39. Under real-world probability measure we assume

$$R(t) = b + \beta R(t-1) + \sigma\varepsilon_t$$

where $0 < \beta < 1$ and ε_t as in the previous section.

R(t) is called an autoregressive process of order 1 (AR(1)).

Remark. In discrete time $\varepsilon_t|\mathcal{F}_{t-1}$ can have any distribution as long as the necessary moments exist.

In discrete time we could take any process for $R(t)$, e.g.

- ARMA(p, q): Autoregressive moving average
- GARCH models

whereas in continuous time one has to be more restrictive in order to obtain solutions.

In order to apply the model to the previous section (?) we choose

$$\lambda(t, R(t-1)) = \lambda \in \mathbb{R} \text{ (fixed)}$$
$$\delta_s = \varepsilon_s$$

then

$$\varphi_t = \exp\left(-\sum_{s=1}^{t} R(s-1) + \tfrac{1}{2}\lambda^2 R(s-1)^2 + \lambda R(s-1)\varepsilon_s\right)$$

Under $\mathbb{P}_0$ we have

$$R(t) = b + (\beta + \lambda\sigma)R(t-1) + \sigma\varepsilon_t^*$$

Lemma 4.40. Under the previous assumptions the distributions of $R(t)$ under the two probability measures are

$$\begin{aligned}
R(t)|\mathcal{F}_s &\overset{P}{\sim} \mathcal{N}\left((1-\beta^{t-s})\frac{b}{1-\beta} + \beta^{t-s}R(s), \frac{1-\beta^{2(t-s)}}{1-\beta^2}\sigma^2\right)\\
R(t)|\mathcal{F}_s &\overset{\mathbb{P}_0}{\sim} \mathcal{N}\left((1-k^{t-s})\frac{b}{1-k} + k^{t-s}R(s), \frac{1-k^{2(t-s)}}{1-k^2}\sigma^2\right)
\end{aligned}$$

with $k = \beta + \lambda\rho$.

Proof. (Sketch)

$$\begin{aligned}
R(t) &= b + \beta R(t-1) + \sigma\varepsilon_t\\
&= b + \beta(b + \beta R(t-2) + \sigma\varepsilon_{t-1}) + \sigma\varepsilon_t \qquad = \ldots\\
&= b(1 + \beta + \beta^2 + \ldots + \beta^{t-s-1}) + \beta^{t-s}R(s) + \sigma\sum_{u=s+1}^{t}\beta^{t-u}\varepsilon_u \quad = b\frac{1-\beta^{t-s}}{1-\beta} + \beta^{t-s}R(s) + S
\end{aligned}$$

where S is a sum of normal random variables, thus

$$S \sim \mathcal{N}\left(0, \sigma^2\sum_{u=s+1}^{t}\beta^{2(t-u)}\right).$$

□

Theorem 4.41. (Vasicek Zero-COupon-Bond Prices)

$$P(s,t) = \exp(A(s,t) - R(s)B(s,t)) \quad (s < t) \tag{27}$$

where

$$\begin{aligned}
A(t-1,t) &= 0\\
B(t-1,t) &= 1\\
A(s,t) &= A(s+1,t) - bB(s+1,t) + \tfrac{1}{2}\sigma^2 B(s+1,t)^2 \quad \text{for } s < t-1\\
B(s,t) &= \frac{1-k^{t-s}}{1-k}
\end{aligned}$$

Remark. Choose $u < s$ then $P(s,t)|\mathcal{F}_u$ is lognormal distributed.

Whenever $P(s,t)$ has the form (27) we say, we have an *affine term structure (ATS).*

What are sufficient model assumptions for obtaining an ATS model? (for ATS in continuous time see [Fil09]

Proof. Under $\mathbb{P}_0$: $R(t) = b + kR(t-1) + \sigma\varepsilon_t^*$. We will proceed with a proof by induction. For $s = t-1$ we have

$$\begin{aligned} P(t-1,t) &= \mathrm{E}_{\mathbb{P}_0}[\exp(R(t-1))|\mathcal{F}_{t-1}] \\ &= \exp(-R(t-1)) \\ &= \exp(0 - 1 \cdot R(t-1)) \end{aligned}$$

For $s < t-1$ we assume the theorem for $s+1$.

$$\begin{aligned} P(s,t) &= \mathrm{E}_{\mathbb{P}_0}\left[\exp(-\sum_{u=s}^{t-1} R(u))|\mathcal{F}_s\right] \\ &= \mathrm{E}_{\mathbb{P}_0}\left[\mathrm{E}_{\mathbb{P}_0}\left[\exp(-\sum_{u=s}^{t-1} R(u))|F_{s+1}\right]\middle|\mathcal{F}_s\right] \\ &= \mathrm{E}_{\mathbb{P}_0}\left[\exp(-R(s))\,\mathrm{E}_{\mathbb{P}_0}\left[\exp(-\sum_{u=s+1}^{t-1} R(u))|F_{s+1}\right]\middle|\mathcal{F}_s\right] \\ &= \mathrm{E}_{\mathbb{P}_0}[\exp(-R(s))P(s+1,t)|\mathcal{F}_s] \\ &= \mathrm{E}_{\mathbb{P}_0}[\exp(-R(s) + A(s+1,t) - R(s+1)B(s+1,t))|\mathcal{F}_s] \\ &= \exp(-R(t) + A(s+1,t) - bB(s+1,t) - kB(s+1,t)R(s)) + \mathrm{E}_{\mathbb{P}_0}\big[\exp(-\sigma B(s+1,t)\varepsilon_{s+1}^*)|\mathcal{F}_s\big] \\ &= \exp(-R(t) + \underbrace{A(s+1,t) - bB(s+1,t)}_{A(s,t)} - \underbrace{kB(s+1,t)R(s)}_{B(s,t)}) + \exp(\tfrac{1}{2}\sigma^2 B^2(s+1,t)) \end{aligned}$$

□

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