Risk Aggregation with Dependence Uncertainty

Carole Bernard

SAA Annual Meeting, online on August 27th, 2021,
Risk Aggregation and Diversification

A key issue in capital adequacy and solvency is to aggregate risks (by summing capital requirements?) and potentially account for diversification (to reduce the total capital?)

\[
\text{std}(X_1 + X_2) = \sqrt{\text{std}(X_1)^2 + \text{std}(X_2)^2 + 2\rho \text{std}(X_1) \text{std}(X_2)}
\]

If \(\rho < 1\), there are “diversification benefits”: \(\text{std}(X_1 + X_2) < \text{std}(X_1) + \text{std}(X_2)\)

This is not the case for instance for Value-at-Risk (but used in regulatory capital requirements).
Risk Aggregation and Diversification

- A key issue in capital adequacy and solvency is to aggregate risks (by summing capital requirements?) and potentially account for diversification (to reduce the total capital?)
- Using the standard deviation to measure the risk of aggregating $X_1$ and $X_2$ with standard deviation $std(X_i)$,

$$std(X_1 + X_2) = \sqrt{std(X_1)^2 + std(X_2)^2 + 2\rho std(X_1)std(X_2)}$$

If $\rho < 1$, there are “diversification benefits”:

$$std(X_1 + X_2) < std(X_1) + std(X_2)$$
A key issue in capital adequacy and solvency is to aggregate risks (by summing capital requirements?) and potentially account for diversification (to reduce the total capital?)

Using the standard deviation to measure the risk of aggregating $X_1$ and $X_2$ with standard deviation $std(X_i)$,

$$std(X_1 + X_2) = \sqrt{std(X_1)^2 + std(X_2)^2 + 2\rho std(X_1)std(X_2)}$$

If $\rho < 1$, there are “diversification benefits”:

$$std(X_1 + X_2) < std(X_1) + std(X_2)$$

This is not the case for instance for Value-at-Risk (but used in regulatory capital requirements).
Motivation on VaR aggregation with dependence uncertainty

**Full** information on marginal distributions:
\[ X_j \sim F_j \]

+ 

**Full** Information on dependence:
(known copula)

\[ \implies \]

\[ \text{VaR}_q (X_1 + X_2 + \ldots + X_d) \] can be computed!
Motivation on VaR aggregation with dependence uncertainty

**Full** information on marginal distributions:
\[ X_j \sim F_j \]

+ 

**Partial** or **no** Information on dependence:
(incomplete information on copula)
\[ \Rightarrow \]

\[ \text{VaR}_q (X_1 + X_2 + \ldots + X_d) \text{ cannot be computed!} \]

Only a range of possible values for \( \text{VaR}_q (X_1 + X_2 + \ldots + X_d) \).
Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of $d$ dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?

Implications:
- Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution "almost completely" or if one knows average pairwise correlation.
Objectives and Findings

- **Model uncertainty on the risk assessment** of an aggregate portfolio: the sum of $d$ dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?

- **Implications:**
  - Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution “almost completely” or if one knows average pairwise correlation.
Acknowledgement of Collaboration (1/2)

with M. Denuit (UCL), X. Jiang (UW), L. Rüschendorf (Freiburg), S. Vanduffel (VUB), J. Yao (VUB), R. Wang (UW):

Acknowledgement of Collaboration (2/2)

More recently with two of my current PhD students, Corrado De Vecchi (VUB) and Rodrigue Kazzi (VUB):


Model Risk

1. **Goal**: Assess the risk of a portfolio sum \( S = \sum_{i=1}^{d} X_i \).
2. Choose a risk measure \( \rho(\cdot) \): variance, Value-at-Risk...
3. “Fit” a multivariate distribution for \( (X_1, X_2, \ldots, X_d) \) and compute \( \rho(S) \)
4. How about model risk? How wrong can we be?
Model Risk

1. Goal: Assess the risk of a portfolio sum \( S = \sum_{i=1}^{d} X_i \).
2. Choose a risk measure \( \rho(\cdot) \): variance, Value-at-Risk...
3. “Fit” a multivariate distribution for \((X_1, X_2, \ldots, X_d)\) and compute \( \rho(S) \).
4. How about model risk? How wrong can we be?

Assume \( \rho(S) = \text{var}(S) \),

\[
\rho^+_{\mathcal{F}} := \sup \left\{ \text{var} \left( \sum_{i=1}^{d} X_i \right) \right\}, \quad \rho^-_{\mathcal{F}} := \inf \left\{ \text{var} \left( \sum_{i=1}^{d} X_i \right) \right\}
\]

where the bounds are taken over all other (joint distributions of) random vectors \((X_1, X_2, \ldots, X_d)\) that “agree” with the available information \(\mathcal{F}\).
Aggregation with dependence uncertainty: Example - Credit Risk

► Marginals known
► Dependence fully unknown

Consider a portfolio of 10,000 loans all having a default probability $p = 0.049$.

<table>
<thead>
<tr>
<th>q</th>
<th>Min $VaR_q$</th>
<th>Max $VaR_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 0.95$</td>
<td>0%</td>
<td>98%</td>
</tr>
<tr>
<td>$q = 0.995$</td>
<td>4.4%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Portfolio models are subject to significant model uncertainty (defaults are rare and correlated events).
Aggregation with dependence uncertainty: Example - Credit Risk

- Marginals known
- Dependence fully unknown

Consider a portfolio of 10,000 loans all having a default probability $p = 0.049$. The default correlation is $\rho = 0.0157$ (for KMV).

<table>
<thead>
<tr>
<th></th>
<th>KMV $\text{VaR}_q$</th>
<th>Min $\text{VaR}_q$</th>
<th>Max $\text{VaR}_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 0.95$</td>
<td>10.1%</td>
<td>0%</td>
<td>98%</td>
</tr>
<tr>
<td>$q = 0.995$</td>
<td>15.1%</td>
<td>4.4%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Portfolio models are subject to significant model uncertainty (defaults are rare and correlated events). Using dependence information is crucial to try to get more “reasonable” bounds.
Outline of the Talk

Part 1: Bounds on Variance
• With full dependence uncertainty
• With partial dependence information on a subset

Part 2: Bounds on Value-at-Risk
• With 2 risks and full dependence uncertainty
• With $d$ risks and full dependence uncertainty
• With partial dependence information on a subset

Part 3: Bounds on Value-at-Risk
• With 2 risks and information on pairwise correlation
• With $d$ risks and information on average correlation
Part I

Bounds on variance
Risk Aggregation and full dependence uncertainty

- Marginals known:
- Dependence fully unknown
- In two dimensions $d = 2$, assessing model risk on variance is linked to the Fréchet-Hoeffding bounds

$$\text{var}(F_1^{-1}(U) + F_2^{-1}(1 - U)) \leq \text{var}(X_1 + X_2) \leq \text{var}(F_1^{-1}(U) + F_2^{-1}(U))$$

- Maximum variance is obtained for the comonotonic scenario:

$$\text{var}(X_1 + X_2 + \ldots + X_d) \leq \text{var}(F_1^{-1}(U) + F_2^{-1}(U) + \ldots + F_d^{-1}(U))$$

- Minimum variance: A challenging problem in $d \geq 3$ dimensions
  - Wang and Wang (2011, JMVA): concept of complete mixability
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
Bounds on variance

Analytical Bounds on Standard Deviation

Consider \( d \) risks \( X_i \) with standard deviation \( \sigma_i \)

\[
0 \leq \text{std}(X_1 + X_2 + \ldots + X_d) \leq \sigma_1 + \sigma_2 + \ldots + \sigma_d.
\]

Example with 20 normal \( N(0,1) \)

\[
0 \leq \text{std}(X_1 + X_2 + \ldots + X_{20}) \leq 20,
\]

in this case, both bounds are sharp and too wide for practical use!

**THUS:** Incorporate information on dependence.
Illustration with 2 risks with marginals $N(0,1)$
Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{ q_{\beta} \leq X_k \leq q_{1-\beta} \}$.
Our assumptions on the cdf of \((X_1, X_2, \ldots, X_d)\)

\[ \mathcal{F} \subset \mathbb{R}^d \text{ ("trusted" or "fixed" area)} \]
\[ \mathcal{U} = \mathbb{R}^d \setminus \mathcal{F} \text{ ("untrusted").} \]

**We assume that we know:**

(i) the marginal distribution \(F_i\) of \(X_i\) on \(\mathbb{R}\) for \(i = 1, 2, \ldots, d\),

(ii) the distribution of \((X_1, X_2, \ldots, X_d) \mid \{(X_1, X_2, \ldots, X_d) \in \mathcal{F}\}\).

(iii) \(p_f := P((X_1, X_2, \ldots, X_d) \in \mathcal{F})\).

► When only marginals are known: \(\mathcal{U} = \mathbb{R}^d\) and \(\mathcal{F} = \emptyset\).

► **Our Goal:** Find bounds on \(\rho(S) := \rho(X_1 + \ldots + X_d)\) when \((X_1, \ldots, X_d)\) satisfy (i), (ii) and (iii).
Example $d = 20$ risks $N(0,1)$

- $(X_1, ..., X_{20})$ independent $N(0,1)$ on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \quad p_f = P((X_1, ..., X_{20}) \in \mathcal{F})$$

(for some $\beta \leq 50\%$) where $q_\gamma$: $\gamma$-quantile of $N(0,1)$.

- $\beta = 0\%$: no uncertainty (20 independent $N(0,1)$).
- $\beta = 50\%$: full uncertainty.

\begin{align*}
\mathcal{F} &= [q_\beta, q_{1-\beta}]^d \\
U &= \emptyset \\
\beta &= 0\% \\
\rho &= 0 \\
U &= \mathbb{R}^d \\
\beta &= 50\% \\
(\rho, 20) &= (0, 20)
\end{align*}

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!
Example $d = 20$ risks $N(0,1)$

- $(X_1, ..., X_{20})$ independent $N(0,1)$ on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \quad p_f = P((X_1, ..., X_{20}) \in \mathcal{F})$$

(for some $\beta \leq 50\%$) where $q_\gamma$: $\gamma$-quantile of $N(0,1)$

- $\beta = 0\%$: no uncertainty (20 independent $N(0,1)$)
- $\beta = 50\%$: full uncertainty

<table>
<thead>
<tr>
<th>$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$</th>
<th>$\mathcal{U} = \emptyset$</th>
<th>$\beta = 0%$</th>
<th>$p_f \approx 98%$</th>
<th>$\beta = 0.05%$</th>
<th>$p_f \approx 82%$</th>
<th>$\mathcal{U} = \mathbb{R}^d$</th>
<th>$\beta = 50%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
<td>$4.47$</td>
<td>$(4.4, 5.65)$</td>
<td>$(3.89, 10.6)$</td>
<td>$(0, 20)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!
Information on the joint distribution

- Can come from a fitted model
- Can come from experts’ opinions
- Dependence “known” on specific scenarios
\[ F_1 = \bigcap_{k=1}^{2} \{ q_\beta \leq X_k \leq q_{1-\beta} \} \]
Illustration with marginals $\mathcal{N}(0,1)$

\[ \mathcal{F}_1 = \text{contour of MVN at } \beta \]

\[ \mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1 \]
Part II

Bounds on Value-at-Risk
VaR aggregation with dependence uncertainty

Our findings

- Maximum Value-at-Risk is not caused by the comonotonic scenario.
- Maximum Value-at-Risk is achieved when the variance is *minimum* in the tail. The RA is then used in the tails only.
- Bounds on Value-at-Risk at high confidence level stay wide even when the trusted area covers 98% of the space!
Risk Aggregation and full dependence uncertainty

Literature review

- Marginals known
- Dependence fully unknown (too wide bounds, all info. ignored)
- Explicit sharp (attainable) bounds
  - $n = 2$ (Makarov (1981), Rüschendorf (1982))
- A challenging problem in $n \geq 3$ dimensions
- Approximate sharp bounds
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
  - Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
“Riskiest” Dependence: maximum $\text{VaR}_q$ in 2 dims?

If $X_1$ and $X_2$ are $U(0,1)$ comonotonic, then

$$\text{VaR}_q(S^c) = \text{VaR}_q(X_1) + \text{VaR}_q(X_2) = 2q.$$
“Riskiest” Dependence: maximum \( \text{VaR}_q \) in 2 dims?

If \( X_1 \) and \( X_2 \) are \( U(0,1) \) comonotonic, then

\[
\text{VaR}_q(S^c) = \text{VaR}_q(X_1) + \text{VaR}_q(X_2) = 2q.
\]

Note that \( \text{TVaR}_q(S^c) = \frac{\int_q^1 2pdp}{1-q} = 1 + q \) (which is also MAX TVaR)
“Riskiest” Dependence: maximum $\text{VaR}_q$ in 2 dims

If $X_1$ and $X_2$ are $U(0,1)$ and antimonotonic in the tail, then $\text{VaR}_q(S^*) = 1 + q$ (which is maximum possible).

\[
\text{VaR}_q(S^*) = 1 + q > \text{VaR}_q(S^c) = 2q
\]

$\Rightarrow$ to maximize $\text{VaR}_q$, the idea is to change the comonotonic dependence such that the sum is constant in the tail.
VaR at level $q$ of the comonotonic sum w.r.t. $q$
VaR at level $q$ of the comonotonic sum w.r.t. $q$

where TVaR (Expected shortfall): $TVaR_q(X) = \frac{1}{1 - q} \int_q^1 VaR_u(X) du$, 

$q \in (0, 1)$
Riskiest Dependence Structure VaR at level $q$

$$S^* \Rightarrow \text{VaR}_q(S^*) = \text{TVaR}_q(S^c)?$$

$$\text{VaR}_q(S^c)$$

$p$  $q$  $1$
Analytic expressions (not sharp)

**Analytical Unconstrained Bounds with** $X_j \sim F_j$

$$A = LTVaR_q(S^c) \leq \text{VaR}_q[X_1 + X_2 + \ldots + X_n] \leq B = TVaR_q(S^c)$$

Approximate sharp bounds:
Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds

Carole Bernard
SAA - AFIR 2021
August 2021 31 / 45
Numerical Results for VaR, 20 risks $N(0,1)$

When marginal distributions are given,

- What is the maximum Value-at-Risk?
- What is the minimum Value-at-Risk?
- A portfolio of 20 risks normally distributed $N(0,1)$. Bounds on $\text{VaR}_q$ (by the rearrangement algorithm applied on each tail)

<table>
<thead>
<tr>
<th>$q$</th>
<th>$(-2.17, 41.3)$</th>
<th>$(-0.035, 71.1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>95%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>99.95%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

▶ Very wide bounds
▶ All dependence information ignored

**Idea:** add information on dependence from a fitted model or from experts’ opinions

**Information on a subset**

VaR bounds when the joint distribution of $(X_1, X_2, ..., X_n)$ is known on a subset of the sample space.
Our assumptions on the cdf of \((X_1, X_2, ..., X_n)\)

\[ \mathcal{F} \subset \mathbb{R}^n \text{ (“trusted” or “fixed” area)} \]
\[ \mathcal{U} = \mathbb{R}^n \setminus \mathcal{F} \text{ (“untrusted”).} \]

We assume that we know:

(i) the marginal distribution \(F_i\) of \(X_i\) on \(\mathbb{R}\) for \(i = 1, 2, ..., n\),

(ii) the distribution of \((X_1, X_2, ..., X_n) | \{ (X_1, X_2, ..., X_n) \in \mathcal{F} \}\).

(iii) \(P((X_1, X_2, ..., X_n) \in \mathcal{F})\)

**Goal:** Find bounds on \(\text{VaR}_q(S) := \text{VaR}_q(X_1 + ... + X_n)\) when \((X_1, ..., X_n)\) satisfy (i), (ii) and (iii).
Numerical Results, 20 correlated $N(0, 1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^n$

<table>
<thead>
<tr>
<th>$\mathcal{F}$</th>
<th>$\mathcal{U} = \emptyset$</th>
<th>$\mathcal{U} = \mathbb{R}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q=95%$</td>
<td>$\beta = 0%$</td>
<td>( -2.17 , 41.3 )</td>
</tr>
<tr>
<td>$q=99.5%$</td>
<td>19.6</td>
<td>( -0.29 , 57.8 )</td>
</tr>
<tr>
<td>$q=99.95%$</td>
<td>25.1</td>
<td>( -0.035 , 71.1 )</td>
</tr>
</tbody>
</table>

- $\mathcal{U} = \emptyset$ : 20 correlated standard normal variables ($\rho = 0.1$).

$$\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.5\%} = 19.6 \quad \text{VaR}_{99.95\%} = 25.1$$
Numerical Results, 20 correlated $N(0, 1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^n$

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{U} = \emptyset$</th>
<th>$p_f \approx 98%$</th>
<th>$p_f \approx 82%$</th>
<th>$\mathcal{U} = \mathbb{R}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q=95%$</td>
<td>$\beta = 0%$</td>
<td>12.5</td>
<td>( 12.2 , 13.3 )</td>
<td>( 10.7 , 27.7 )</td>
</tr>
<tr>
<td>$q=99.5%$</td>
<td>$\beta = 0.05%$</td>
<td>19.6</td>
<td>( 19.1 , 31.4 )</td>
<td>( 16.9 , 57.8 )</td>
</tr>
<tr>
<td>$q=99.95%$</td>
<td>$\beta = 0.5%$</td>
<td>25.1</td>
<td>( 24.2 , 71.1 )</td>
<td>( 21.5 , 71.1 )</td>
</tr>
</tbody>
</table>

- $\mathcal{U} = \emptyset$ : 20 correlated standard normal variables ($\rho = 0.1$).

\[
\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.5\%} = 19.6 \quad \text{VaR}_{99.95\%} = 25.1
\]

- The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.

- For VaR at high probability levels ($q = 99.95\%$), despite all the added information on dependence, the bounds are still wide!
The Basel Committee (2013) insists that a **desired objective of a Solvency framework concerns comparability**: 

"Two banks with portfolios having identical risk profiles apply the frameworks rules and arrive at the same amount of risk-weighted assets, and two banks with different risk profiles should produce risk numbers that are different proportionally to the differences in risk"
How does correlation impact Value-at-Risk bounds?

Carole Bernard 1, 2  Corrado De Vecchi 1  Steven Vanduffel 1

1 Vrije Universiteit Brussel

2 Grenoble Ecole de Management

August 2021
Upper bound for $\text{VaR}_q^+(X_1 + X_2)$

Assume $X_i$ has marginal cdf $F_i$ and $C$ denotes the copula for $(X_1, X_2)$

$\delta(C, F_1, F_2) = \text{Spearman’s rho, Kendall’s tau or Pearson correlation.}$

$\overline{\text{VaR}}_q^d := \sup \text{VaR}_q^+(X_1 + X_2)$
subject to $X_j \sim F_j, \ j = 1, 2$
$\delta(C, F_1, F_2) = d.$

(1)

Unconstrained problem:

$\overline{\text{VaR}}_q := \sup \text{VaR}_q^+(X_1 + X_2)$
subject to $X_j \sim F_j, \ j = 1, 2.$

(2)
Upper bound for $\text{VaR}_q^+(X_1 + X_2)$: copulas

Given $q \in (0, 1)$, consider the squares $[0, q]^2$ and $[q, 1]^2$.

![Graphs showing supports of $C_{\min}$ (left) and $C_{\max}$ (right) for $q = 0.8$.]

**Figure:** Supports of $C_{\min}$ (left) and $C_{\max}$ (right) for $q = 0.8$.

**Definition:**

Given $q \in (0, 1)$, let $\delta$ be a measure of dependence (Kendall’s tau, Spearman’s rho or Pearson correlation), $F_1$ and $F_2$ two c.d.f, we define

\[
\delta_{\min} = \delta(C_{\min}, F_1, F_2)
\]

\[
\delta_{\max} = \delta(C_{\max}, F_1, F_2)
\]
Upper bound for $\text{VaR}_q^+(X_1 + X_2)$: results

\[ \text{VaR}_q^d \equiv \sup \text{VaR}_q^+(X_1 + X_2) \]

subject to

\[ X_j \sim F_j, \quad j = 1, 2 \]
\[ \delta(C, F_1, F_2) = d. \]  

(3)

Theorem:

Given $q \in (0, 1)$, let $\delta$ be a measure of dependence (Kendall’s tau, Spearman’s rho or Pearson correlation), $F_1$ and $F_2$ two c.d.f.

For every $d \in [\delta_{\text{min}}, \delta_{\text{max}}]$ it holds that

\[ \text{VaR}_q^d = \text{VaR}_q. \]  

(4)

and the upper bound is attained.

Note $d \in [\delta_{\text{min}}, \delta_{\text{max}}] \implies \text{constraint is redundant}$
Upper bound for $\text{VaR}_q^+(X_1 + X_2)$: results

- Fix $\delta$ and $d$. If $d \in [\delta_{\text{min}}, \delta_{\text{max}}]$, then $M(d) = \overline{M}$.
- $[\delta_{\text{min}}, \delta_{\text{max}}]$ is easy to compute.
- for $q \geq 0.95$, $[\delta_{\text{min}}, \delta_{\text{max}}]$ almost covers the range of values for $\delta$.

A: $X_1 \sim \text{Gamma}(2, 3)$,  
$X_2 \sim \text{Lognormal}(2, 1)$,

B: $X_i \sim N(0, 1), i = 1, 2$. 

![Graphs showing the behavior of $\text{corr}_{\text{max}}$ and $\text{corr}_{\text{min}}$ as $q$ varies.](image1)

![Graphs showing the behavior of $\text{corr}_{\text{max}}$ and $\text{corr}_{\text{min}}$ as $q$ varies.](image2)
Interval $[\delta_{\min}, \delta_{\max}]$

1. $\delta_{\min}$ and $\delta_{\max}$ are very easy to compute:
   - Spearman’s rho:
     \[
     \rho_{\min} = -6q(q - 1) - 1 \quad \text{and} \quad \rho_{\max} = 1 - 2(1 - q)^3.
     \]
   - Kendall’s tau:
     \[
     \tau_{\min} = -4q(q - 1) - 1 \quad \text{and} \quad \tau_{\max} = -2(q - 1)^2 + 1.
     \]

2. For $q \approx 1$, $[\delta_{\min}, \delta_{\max}]$ almost covers the range of values of $\delta$.

Table: $\delta =$Spearman’s rho, range $[-1, 1]$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\delta_{\min}$</th>
<th>$\delta_{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>95.0%</td>
<td>-0.715</td>
<td>0.999</td>
</tr>
<tr>
<td>99.0%</td>
<td>-0.941</td>
<td>0.999</td>
</tr>
<tr>
<td>99.5%</td>
<td>-0.970</td>
<td>0.999</td>
</tr>
</tbody>
</table>
RVaR bounds with \( n \) risks

**Average correlation:** given a portfolio \( \mathbf{X} = (X_1, \ldots, X_n) \), the average correlation of \( \mathbf{X} \), \( \text{acorr} (\mathbf{X}) \), is defined as

\[
\text{acorr} (\mathbf{X}) = \frac{\sum_{i \neq j} \text{corr}(X_i, X_j) \text{std}(X_i) \text{std}(X_j)}{\sum_{i \neq j} \text{std}(X_i) \text{std}(X_j)}.
\]

(5)

**Range Value-at-Risk:**

\[
\text{RVaR}_{\alpha, \beta}(\mathbf{X}) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_\gamma(\mathbf{X}) d\gamma, \quad 0 < \alpha < \beta < 1.
\]

(6)

**Problems:**

\[
\sup \left\{ \inf \left\{ \text{RVaR}_{\alpha, \beta}(S) \left| S = \sum_{i=1}^{n} X_i, \ X_i \sim F_i \right. \right\} \right\}.
\]

(7)

\[
\sup \left\{ \inf \left\{ \text{RVaR}_{\alpha, \beta}(S) \left| S = \sum_{i=1}^{n} X_i, \ X_i \sim F_i, \ \text{acorr} (\mathbf{X}) \leq d \right. \right\} \right\}.
\]

(8)
RVaR bounds

1. No dependence information: given $X_i \sim F_i$,

   $$A(\beta) \leq \text{RVaR}_{\alpha,\beta}(S) \leq B(\alpha). \quad (9)$$

2. Average correlation constraint: given $X_i \sim F_i$ and $\text{acorr}(X) \leq d$,

   $$l(\beta) \leq \text{RVaR}_{\alpha,\beta}(S) \leq u(\alpha). \quad (10)$$

- Sharpness: tail mixability (sufficient).
- VaR and TVaR bounds as special cases.
- If $d \geq \max (c(\alpha), c(\beta))$, then $l(\beta) = A(\beta)$ and $u(\alpha) = B(\alpha) \implies \text{constraint is redundant.}$
Conclusions

Pitfall to avoid:

- Knowledge of dependence measure (such as a correlation coefficient or an average correlation) may not help to improve a risk measure worst-case scenario.

With Value-at-Risk, only tail information helps. In the paper, we also show that

- Knowledge (or realistic assumption) regarding tail dependence is more effective.
Thank you for listening!