SAA Working Group on Yield Curves
First Results

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EPFL and Swiss Finance Institute

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SAA Working Group on Yield Curves

Goals:
- Provide technical input to FINMA-AG Zinskurven
- Application of Kernel Ridge (KR) method developed in Filipović–Pelger–Ye (2022) “Stripping the discount curve – a robust machine learning approach” to CHF, EUR, USD, GBP, JPY
- Explore data sources according to criteria availability, quality, completeness, cost
- Further development of KR method towards multi-currency learning

Organisation:
- Lead by Lutz Wilhelmy and Damir Filipović
- WG members from CH industry (Baloise, Generali, Mobiliar, Swiss Life, Swiss Re, Zurich) and academia (EPFL)
- Kick-off in June 2022
Principles of Yield Curve Estimation

- Simple and fast to implement
- Transparent and reproducible
- Data-driven
- Precise representation of the term structure, taking into account all market signals
- Robust to outliers and data selection choices
- Flexible for integration of external views: exogenous points, choice of weights
- Consistent with finance principles
Methods in scope

- SNB Nelson–Siegel–Svensson (2002): NSS with parameter constraints to match overnight rate (since 2021 SARON 1M-swap)
- Smith–Wilson (2001): interpolation-extrapolation method, Solvency II standard, used for SST since 2012 based on SNB NSS
Outline

1 KR Method

2 Empirical study
Outline

1. KR Method
2. Empirical study
Ingredients

- **Unobserved discount curve** $g(x) =$ fundamental value of a non-defaultable zero-coupon bond with time to maturity $x$

- Observed: $M$ fixed income securities with
  - cash flow dates $0 < x_1 < \cdots < x_N$
  - $M \times N$ cash flow matrix $C$
  - noisy ex-coupon prices $P = (P_1, \ldots, P_M)^\top$

- No-arbitrage pricing relation:
  $$P_i = C_i g(x) + \epsilon_i,$$
  where $x = (x_1, \ldots, x_N)^\top$ and $g(x) = (g(x_1), \ldots, g(x_N))^\top$

- $\epsilon_i$: deviations from fundamental value, due to market imperfections (no deep, liquid, transparent market) and data errors
Estimation problem

**Problem: Minimize pricing errors** for some exogenous weights $\omega_i$:

$$\min_g \left\{ \sum_{i=1}^{M} \omega_i (P_i - C_i g(x))^2 \right\}$$

- Observe only $M \approx 25$ bonds, need to estimate $N \approx 15,000$ (40 years $\times$ 365 days) discount bond prices
- Any estimation approach imposes regularizing assumptions to limit the number of parameters
- Existing approaches ad-hoc assumptions $\Rightarrow$ misspecified form

**KR approach: Smoothness regularization**

- Limits to arbitrage require a sufficiently smooth curve, as large sudden changes imply risk-free extreme payoffs
Smooth discount curves

General measure of smoothness for functions

\[ \|g\|_{\alpha,\delta} = \left( \int_0^{\infty} (\delta g'(x)^2 + (1 - \delta) g''(x)^2) e^{\alpha x} \, dx \right)^{\frac{1}{2}} \]

- **Curvature** $g''(x)^2$: penalizing avoids kinks
- **Tension** $g'(x)^2$: penalizing avoids oscillations
- **Maturity weight** $\alpha \geq 0 \Rightarrow$ corresponds to infinite-maturity yield
- **Tension parameter** $\delta \in [0, 1)$ balances tension and curvature

⇒ Work with extremely large hypothesis space of discount curves given by the set $G_{\alpha,\delta}$ of twice differentiable functions $g : [0, \infty) \rightarrow \mathbb{R}$ with $g(0) = 1$ and finite smoothness measure $\|g\|_{\alpha,\delta} < \infty$
Fundamental estimation problem

Fundamental optimization problem:

\[
\min_{g \in G_{\alpha, \delta}} \left\{ \sum_{i=1}^{M} \omega_i (P_i - C_i g(x))^2 + \lambda \|g\|_{\alpha, \delta}^2 \right\}
\]

- **Smoothness parameter** \( \lambda > 0 \): Trade-off between pricing errors and smoothness
- **Exogenous weights** \( 0 < \omega_i \leq \infty \) \( (\omega_i = \infty \) is exact pricing\): we set \( \omega_i \) to duration weights \( \Rightarrow \) approximate yield fitting
- **Problem completely determined up to the three parameters** \( \alpha, \delta, \lambda \) selected empirically via cross-validation to minimize pricing errors out-of-sample \( \Rightarrow \) fully data-driven.
Kernel Ridge (KR) solution

The KR solution to fundamental problem (1) is given by:

\[ \hat{g}(x) = 1 + \sum_{j=1}^{N} k(x, x_j) \beta_j, \quad \text{where} \quad \beta = C^T (CKC^T + \Lambda)^{-1} (P - C1), \]

for $N \times N$-kernel matrix $K_{ij} = k(x_i, x_j)$, and $\Lambda = \text{diag} (\lambda/\omega_1, \ldots, \lambda/\omega_M)$

- Simple closed-form solution, easy to implement
- Basis functions $k(., x_j)$ are determined by smoothness measure
- Discount bonds are portfolios of coupon bonds $\Rightarrow$ Immunization
- Nelson–Siegel–Svensson and Smith–Wilson discount curves are special cases of KR framework for specific parameter choices.
Special curves: Nelson–Siegel–Svensson

Nelson–Siegel–Svensson (NSS) assume a parametric forward curve

\[ f_{NSS}(x) = \gamma_0 + \gamma_1 e^{-\frac{x}{\tau_1}} + \gamma_2 \frac{x}{\tau_1} e^{-\frac{x}{\tau_1}} + \gamma_3 \frac{x}{\tau_2} e^{-\frac{x}{\tau_2}} \]

for real parameters \(\gamma_0, \gamma_1, \gamma_2, \gamma_3\) and \(\tau_1, \tau_2 > 0\).

Lemma 1.1.

The NSS curve \(g_{NSS}(x) = e^{-\int_0^x f_{NSS}(t) \, dt}\) lies in \(G_{\alpha,\delta}\), if \(\alpha < 2\gamma_0\).
Special curves: Smith–Wilson

Smith–Wilson assume discount curves of the form

\[ g_{SW}(x) = e^{-y_\infty x} g_0(x), \quad y_\infty := \log(1 + UFR), \]

for some \( g_0 \in G_{0,1/2} \) and ultimate forward rate \( UFR > 0 \).

- Assume exact pricing up to last liquid point (minimal regularity)
- Insurance industry standard in Europe
- Used in the regulatory Solvency II framework

Lemma 1.2.

The Smith–Wilson curve \( g_{SW} \) lies in \( G_{\alpha,\delta} \), if \( \alpha < 2y_\infty \).
Bayesian perspective and distribution theory

Assume $g$ is a Gaussian process with prior distribution

$$g(x) \sim \mathcal{N} \left( m(x), k(x, x^\top) \right),$$

with pricing errors $\epsilon \sim \mathcal{N}(0, \Sigma^\epsilon)$ for $\Sigma^\epsilon = \text{diag}(\sigma_1^2, \ldots, \sigma_M^2)$.

**Theorem 1.3 (Bayesian perspective).**

If the prior mean function $m(x) = 1$ and pricing error variance $\sigma_i^2 = \lambda/\omega_i$, then

1. the posterior mean function equals the KR estimated discount curve,
2. the posterior distribution is Gaussian with known posterior variance.

$\Rightarrow$ We obtain a confidence range for the discount curve and securities
Outline

1. KR Method

2. Empirical study
Data, estimation and evaluation

CH Confederation bonds:
- CH Confederation bond data from SNB public source
- Daily ex-dividend (clean) mid-prices (adjusted for AI)
- Sampling period: January 2010 to June 2022 (150 months)
- Total of 22 issues of Confederation bonds

Estimation and evaluation:
- Estimation without bonds maturing in less than 3M
- In-sample evaluation with all bonds
- Cross-sectional out-of-sample with LOO cross-validation
- Root-mean-squared errors (RMSE) for yields and relative prices
Data: maturity ranges

- Figure shows the time to maturity of the data. Red: maximum
- Unequal maturity distribution: long maturities underrepresented
- Unbalanced panel: > 40 years only available after July 2014
Cross-validation for hyper-parameters $\alpha$, $\delta$, $\lambda$

- Figure shows average cross-validation YTM fitting error (in bps)
- Optimal values (baseline choice): $\lambda = 10$, $\alpha = 0.02$, $\delta = 0$
- Results are robust to the choice of hyper-parameters
Illustration: yield curve estimates as fct of parameters

- **Representative example day:** 2016-07-29
- **Effect of $\lambda$:** less curvature $\Rightarrow$ bias-variance tradeoff
- **Effect of $\alpha$:** only affects long maturities $\Rightarrow \alpha = $ infinite-maturity yield

$\Rightarrow$ Extrapolation is a choice and not verifiable on observed data
Illustration: yield curve estimates as function of parameters

- Representative example day: 2020-04-30
- Effect of $\lambda$: less curvature $\Rightarrow$ bias-variance tradeoff
- Effect of $\alpha$: only affects long maturities $\Rightarrow$ $\alpha = \text{infinite-maturity yield}$

$\Rightarrow$ Extrapolation is a choice and not verifiable on observed data
Benchmark models

- Nelson–Siegel–Svensson (NSS): non-convex optimization $\Rightarrow$ not (easily) reproducible
- SNB Nelson–Siegel–Svensson: NSS with parameter constraints
- SST curves (since 2021): Smith–Wilson method based on SNB NSS
Average in-sample pricing errors for different maturities

- In-sample evaluation with all bonds
- KR dominates all benchmark methods along all maturities
- KR has smallest yield and pricing errors for all bonds, also over time . . .
Time series for in-sample YTM RMSE per bucket
Representative example day: 2016-07-29

- NSS curves not flexible and excessive curvature in the short end
- SST curve biased by UFR (left panel)
- 99% confidence intervals wider for maturities with more dispersed or less observed prices
Illustration: yield curve estimates of different methods

- Representative example day: 2016-07-29
- NSS curves not flexible and excessive curvature in the short end
- SST curve biased by UFR (left panel)
- 99% confidence intervals wider for maturities with more dispersed or less observed prices
Short and long maturity yield estimates over time

5Y and 10Y yield estimates: similar volatility
Short and long maturity yield estimates over time

15Y and 30Y yield estimates: similar volatility
Short and long maturity yield estimates over time

50Y and 100Y yield estimates: extrapolations can be very volatile \(\Rightarrow\) exogenous points necessary
Conclusion and outlook

- KR method satisfies all principles of yield curve estimation
- KR method dominates NSS-SNB and SST curves: easily reproducible and most precise representation of the term structure
- Extrapolation to 50Y and beyond: requires exogenous input
- E.g., multi-curve learning CHF, EUR, USD, GBP, JPY, learn about CHF curve from long maturities of other currencies (e.g., Austria 100Y Government Bond) ⇒ ongoing research
Backup: list of WG members

- Lutz Wilhelmy (Swiss Re)
- Damir Filipović (EPFL)
- Nicolas Camenzind (EPFL)
- Andreas Lutz (Baloise)
- Dominik Stich (Baloise)
- Philipp Keller (Generali)
- Oliver Strub (Mobiliar)
- Urs Müller (Swiss Life)
- Tsunehiro Tsujimoto (Swiss Re)
- Jozef Minar (Zurich)