Value-at-Risk vs Expected Shortfall: A Financial Perspective

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Objective of the presentation

- The two risk measures that are most widely used as the basis for economic solvency regimes are Value-at-Risk (VaR) in Solvency II and Expected Shortfall (ES) in SST
- ES was has been generally viewed as being "better" than VaR from a theoretical perspective because it
  - $\rightarrow$ takes a policyholder perspective (is not blind to the tail and disallows build up of uncontrolled loss peaks)
  - $\rightarrow$ gives credit for diversification (is coherent)

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- In this presentation we challenge the view that ES takes a policyholder perspective
  - $\rightarrow$ This complements the current discussion on ES vs. VaR which is based exclusively on a criticism of statistical properties of ES (for an overview of this discussion see [3])

Basic question in a capital adequacy framework

Starting point: At $t = 0$ a financial institution selects a portfolio of *assets* and *liabilities* and at $t = T$ assets are liquidated and liabilities repaid

- $\longrightarrow$ Liability holders worry that the institution may *default* at time T, i.e. that *capital* (= "assets minus liabilities") may become negative at $t = T$, ...
- $\longrightarrow$ ... but they are also unwilling to bear the costs of fully eliminating the risk of default and have to settle for some acceptable level of security

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Key question for regulators: what is an acceptable level of security for policyholder liabilities, i.e. when should an insurer be deemed to be adequately capitalized?

Testing for capital adequacy: acceptance sets

Capital position of insurers, i.e. assets minus liabilities, at time T are random variables $X : \Omega \to \mathbb{R}$ defined (for simplicity) on finite state space $\Omega := \{\omega_1, \ldots, \omega_n\}$.
$\mathscr{X}$ denotes the vector space of all possible capital positions

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Regulators subject insurers to a *capital adequacy test* by checking whether their capital positions belong to an *acceptance set* $\mathscr{A} \subset \mathscr{X}$ satisfying two minimal requirements:

→ *Non-triviality:* $\emptyset \neq \mathscr{A} \neq \mathscr{X}$

→ *Monotonicity:* $X \in \mathscr{A}$ and $Y \geq X$ imply $Y \in \mathscr{A}$

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Remark

1. *Because they capture diversification effects, convex acceptance sets or coherent acceptance sets (acceptance sets that are convex cones) are of particular interest*
2. *We use interchangeably: acceptance set, capital adequacy test, acceptability criterion*

The simplest acceptability criterium: scenario testing

The simplest acceptance criterion is testing whether an insurer can meet its obligations on a pre-specified set of states of the world $A \subset \Omega$. The corresponding acceptance sets are called of SPAN-*type* and given by

$$\mathrm{SPAN}(A) := \{X \in \mathscr{X} \; ; X(\omega) \geq 0 \text{ for every } \omega \in A\} \; .$$

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Remark

1. SPAN *stands for Standard Portfolio ANalysis.*
2. $\mathrm{SPAN}(A)$ *is a closed, coherent acceptance set.*
3. *In the extreme case $A = \Omega$, the set $\mathrm{SPAN}(A)$ coincides with the set of positive random variables, i.e. an insurer would be required to be able to pay claims in every state of the world!*

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$$\mathscr{A}_\alpha := \{X \in \mathscr{X} \, ; \, \mathbb{P}(X < 0) \leq \alpha\} = \{X \in \mathscr{X} \, ; \, \mathrm{VaR}_\alpha(X) \leq 0\} \, ,$$

where

$$\mathrm{VaR}_\alpha(X) := \inf \{m \in \mathbb{R} \, ; \, \mathbb{P}(X + m < 0) \leq \alpha\} \, .$$

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The *Expected Shortfall acceptance set* at the level $0 < \alpha < 1$ is closed and coherent and defined by

$$\mathscr{A}^\alpha := \{X \in \mathscr{X}\,;\mathrm{ES}_\alpha(X) \leq 0\}\,,$$

where

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,d\beta\,.$$

Standard properties of VaR_α and ES_α

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(a) VaR_α and ES_α are *cash-additive*, i.e. if ρ is either VaR_α or ES_α, then

$$\rho(X+m) = \rho(X) - m \qquad \text{for } X \in \mathscr{X} \text{ and } m \in \mathbb{R}$$

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(c) VaR_α and ES_α are *positively homogeneous*, i.e. if ρ is either VaR_α or ES_α, then

$$\rho(\lambda X) = \lambda \rho(X) \qquad \text{for } X \in \mathscr{X} \text{ and } \lambda \geq 0$$

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(d) ES_α is *subadditive*, i.e.

$$\mathrm{ES}_\alpha(X+Y) \leq \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(Y) \qquad \text{for } X, Y \in \mathscr{X}$$

Capital adequacy tests in terms of available and required capital

If X_0 is the capital position at time 0 and ΔX is the *profit* for the period $[0, 1]$, then

$$\begin{aligned} X &= X_0 + \Delta X \\ &= X_0 + \overline{\Delta X} + R \end{aligned}$$

where $R := \Delta X - \overline{\Delta X}$ is the deviation around *expected profit* $\overline{\Delta X}$.

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If $\rho : \mathscr{X} \to \mathbb{R}$ is either VaR_α or ES_α we have

$$\begin{aligned} \rho(X) \le 0 \quad &\iff \quad \rho(\Delta X) \le X_0 \\ &\iff \quad \underbrace{\rho(R) - \overline{\Delta X}}_{\textit{required capital}} \le \underbrace{X_0}_{\textit{available capital}} \end{aligned}$$

Assume $X = A - L$ and $Y = A' - L$ are the capital positions of two insurers with identical liabilities and possibly different assets. The respective *payoffs to policyholders* are

$$P_X := L - D_X \quad \text{and} \qquad P_Y := L - D_Y$$

where the respective *insurers' options to default* D_X and D_Y are defined by

$$D_X := \max\{-X, 0\} \quad \text{and} \qquad D_Y := \max\{-Y, 0\}$$

Clearly,

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Motivating example

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It is reasonable to expect that policyholders are indifferent to having their liabilities with the first or with the second insurer since in both instances they get exactly the same amounts in the same states of the world

$\rightarrow$ X and Y should be either both acceptable or both unacceptable!

Surplus invariance

Definition ([5])

An acceptance set $\mathscr{A} \subset \mathscr{X}$ is said to be *surplus invariant*, if

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→ The name surplus invariance comes from the decomposition

$$X = S_X - D_X$$

where $S_X := \max\{X, 0\}$ is the *surplus*. An acceptance set is surplus invariant if acceptability does not depend on the surplus but only on the default option.

VaR_α acceptability <u>is</u> surplus invariant

Proposition

Then the acceptance set $\mathscr{A}_\alpha$ is surplus invariant, i.e.

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$[\ \mathbb{P}(Y < 0) = \mathbb{P}(D_Y > 0) = \mathbb{P}(D_X > 0) = \mathbb{P}(X < 0) \leq \alpha\ ]$

This does not invalidate the fundamental criticism of VaR_α:

- $\rightarrow$ As long as $\mathbb{P}(X < 0) \leq \alpha$ holds it is blind to what happens on $\{\omega \in \Omega\,; X(\omega) < 0\}$ and, therefore, allows the build up of uncontrolled loss peaks on that set!
- $\rightarrow$ It does not capture diversification!

ES_α acceptability is not surplus invariant

Proposition ([4])

Let $X \notin \mathscr{A}^\alpha$. The following statements are equivalent:

(a) *There exists $Y \in \mathscr{A}^\alpha$ such that $D_X = D_Y$;*

(b) $\mathbb{P}(X < 0) < \alpha$

(c) $X \in \mathscr{A}_\beta$ *for some* $\beta \in (0, \alpha)$.

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This situation arises in the region that distinguishes Solvency II (based on $\mathrm{VaR}_{0.5\%}$) and SST (based on $\mathrm{ES}_{1\%}$):

→ If $\mathrm{VaR}_{0.5\%}(X) \leq 0$, i.e. X is accepted under Solvency II, and $\mathrm{ES}_{1\%}(X) > 0$, i.e. X is rejected under SST, then we find $Y \in \mathscr{X}$ such that $D_Y = D_X$ and $\mathrm{ES}_{1\%}(Y) \leq 0$, i.e. Y is accepted under SST.

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Theorem ([5])

The only coherent surplus invariant acceptance sets are those of SPAN*-type. The only law- and surplus-invariant coherent acceptance set is the set of random variables that are everywhere positive.*

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The only coherent surplus invariant acceptance sets are those of SPAN*-type. The only law- and surplus-invariant coherent acceptance set is the set of random variables that are everywhere positive.*

- $\rightarrow$ The only law-invariant, coherent acceptability criterion that is surplus invariant is the most conservative one: the insurer must be solvent in all states of the world!
- $\rightarrow$ All other coherent surplus invariant criteria are of the form $\mathrm{SPAN}(A)$ and suffer from a similar shortcoming as VaR_α: they are blind to what happens on A^c and, therefore, allow build up of uncontrolled loss peaks on that set!

Conclusion

Multiple competing requirements			
	Captures diversification	Controls loss peaks	Is surplus invariant
SPAN	✔	✘	✔
VaR	✘	✘	✔
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THANK YOU FOR YOUR ATTENTION!

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