Variable Annuities with Guaranteed Minimum Withdrawal Benefits

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SAA Annual Meeting 2016 in Fribourg
Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
Variable annuities with GMxB

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- Variable annuities with protection features have gained popularity
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- GMDB: Guaranteed minimum death benefits
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- GMDB: Guaranteed minimum death benefits
- GMAB: Guaranteed minimum accumulation benefits
- GMIB: Guaranteed minimum income benefits
- GMWB: Guaranteed minimum withdrawal benefits

(See Gerold Studer's talk at the SAA Annual Meeting 2010 for an overview)
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Global development of Variable Annuities

- **USA**
  - New guarantees for pension plans “401 k” (volume 5-7 bn USD)

- **Canada**
  - First mover in terms of VA regulation

- **Japan**
  - Growing business since 2002
  - Word leader in terms of new business
  - Few “Variable Annuities” products in the market
  - But many projects for new products going on
  - Start looking at VA
Increasing popularity of GMWB with high water mark features (ratchets)

Description on the Vanguard Group website of one of their VA+GMWB
What’s the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
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  - how insurance companies should set the fee structure
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- Derive a hedging strategy
The financial market

Dynamics of an underlying mutual fund

\[ dS_t = S_t(\mu dt + \sigma dW_t) = S_t(rd_t + \sigma dB_t) \]
The financial market

Dynamics of an underlying mutual fund

\[ dS_t = S_t (\mu dt + \sigma dW_t) = S_t (rdt + \sigma dB_t) \]

where

- \( W \) is a Brownian motion under the real world probability measure \( \mathbb{P} \)
- \( \mu \) is the expected return rate
- \( \sigma \) is the volatility
- \( B \) is a Brownian motion under the risk-neutral probability measure \( \mathbb{Q} \)
- \( r \) is a constant risk-free rate
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Market filtration

\[ \mathbb{F} = (\mathcal{F}_t) \quad \text{generated by } (S_t) \]
The contract

- Fixed maturity $T$, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
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  keeps $c_2$ as a commission and charges fees at rate $qA_t$
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Annuity account

$$dA_t = ((r - q)A_t - w_t)dt + \sigma A_t dB_t, \quad A_0 = c_1$$

with absorption at 0
Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $G = (G_t)$
Assume: $F$ and $G$ are independent under $Q$
Information of the policy holder: $H = F \lor G$
Withdrawal

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  - For $\alpha M_t < w_t \leq \beta M_t$, fees are charged at a higher rate $p_2$
Withdrawal

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Information of the policy holder: $H = F \vee G$

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- **Guaranteed withdrawal** If the account hits 0, withdrawal is still allowed.
Withdrawal

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**Income rate**

$$f(w_t, M_t) = (1 - p_1) \min(w_t, \alpha M_t) + (1 - p_2) \max(w_t - \alpha M_t, 0), \quad w_t \leq \beta M_t$$
The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$. 
Surrender

- The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$

- At time $\theta$ the issuer charges a penalty of $k(\theta)A_\theta$, returns $(1 - k(\theta))A_\theta$ to the holder and terminates the contract,

where $k : [0, T] \rightarrow [0, 1]$ is a surrender penalty function
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- Typically, $k(T) = 0$
Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$.
Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$
- Worst expected cost of the payments faced by the issuer

$$D(A_0) = \sup_{w, \theta} \mathbb{E}^Q \int_0^\theta 1\{A_t = 0\} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta$$

$$-\mathbb{E}^Q \int_0^\theta 1\{A_t > 0\} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds$$

One has $A_0 + D(A_0) = \mathbb{E}(A_0)$ for $\mathbb{E}(A_0) = \sup_{w, \theta} \mathbb{E}^Q \left\{ \int_0^\theta 1\{A_t = 0\} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta \right\}$.

Worst case policy holder behavior is not actual policy holder behavior!

The policy is correctly priced (from the issuer's perspective) if $D(c_1) = c_2 \iff \mathbb{E}(c_1) = c_1 + c_2$.

Good news: $\mathbb{E}(A_0)$ is attained for $w, \theta$ only depending on market information $F$.

Bad news: The optimization problem $\mathbb{E}(A_0)$ is not Markovian.
Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$.
- **Worst expected cost** of the payments faced by the issuer:

$$D(A_0) = \sup_{w,\theta} \mathbb{E}^Q \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta$$

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- One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w,\theta} \mathbb{E}^Q \left[ \int_0^\theta e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta)) A_\theta \right]$$

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- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$
- **Worst expected cost** of the payments faced by the issuer

\[
D(A_0) = \sup_{w,\theta} \mathbb{E}^Q \left[ \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta \right. \\
\left. - \mathbb{E}^Q \int_0^\theta 1_{\{A_t>0\}} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds \right]
\]

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- **Good news:** $E(A_0)$ is attained for $w, \theta$ only depending on market information $\mathcal{F}$
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$$D(c_1) = c_2 \iff E(c_1) = c_1 + c_2$$

- **Good news:** $E(A_0)$ is attained for $w, \theta$ only depending on market information $\mathbb{F}$
- **Bad news:** The optimization problem $E(A_0)$ is not Markovian
Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, $w$, define

$$dA_{s}^{t,a,w} = ((r - m)A_{s}^{t,a,w} - w_{s})ds + \sigma A_{s}^{t,a,w} dB_{s}, \quad A_{t}^{t,a,w} = a$$

$$M_{s}^{t,a,g,w} = m \vee \sup_{t \leq u \leq s} A_{u}^{t,a,m,w}$$

$$dZ_{s}^{t,z,w} = w_{s}ds, \quad Z_{t}^{t,z,w} = z$$
Adding more state variables

For given \(0 \leq t \leq T, 0 \leq a \leq m, 0 \leq z \leq \gamma m, w\), define

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dA_s^{t,a,w} = ((r - m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a
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M_s^{t,a,g,w} = m \vee \sup_{t \leq u \leq s} A_u^{t,a,m,w}
\]

\[
dZ_s^{t,z,w} = w_s ds, \quad Z_t^{t,z,w} = z
\]

A "non-standard" standard stochastic control problem

\[
V(t, a, m, z) = \sup_{w, \theta} \mathbb{E}^Q \left[ \int_t^T e^{-r(s-t)} f(w_s, M_s^{t,a,m,w}) ds + e^{-r(\theta-t)} (1 - k(\theta)) A_\theta^{t,a,w} \right]
\]

where \(w\) and \(\theta\) are adapted to \(\mathcal{F}_s^t = \sigma(B_s - B_t), s \in [t, T]\)
Adding more state variables

For given \(0 \leq t \leq T, 0 \leq a \leq m, 0 \leq z \leq \gamma m, w\), define

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dA_{s}^{t,a,w} = ((r - m)A_{s}^{t,a,w} - w_{s})ds + \sigma A_{s}^{t,a,w} dB_{s}, \quad A_{t}^{t,a,w} = a
\]

\[
M_{s}^{t,a,g,w} = m \lor \sup_{t \leq u \leq s} A_{u}^{t,a,m,w}
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dZ_{s}^{t,z,w} = w_{s}ds, \quad Z_{t}^{t,z,w} = z
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\]

where \(w\) and \(\theta\) are adapted to \(\mathcal{F}_{s} = \sigma(B_{s} - B_{t}), s \in [t, T]\)

One has \(V(0, a, a, 0) = E(a)\)
The HJB equation

**Theorem**

\( V(t, a, m, z) \) is a viscosity solution of

\[
\begin{align*}
\min(-v_t - H(a, m, v, v_a, v_z, v_{aa}), v - (1 - k)a) &= 0 \text{ for } 0 < a < m, 0 \leq z < \gamma m \\
\min(-v_t - H_0(a, v, v_a, v_{aa}), v - (1 - k)a) &= 0 \text{ for } 0 < a < m, z = \gamma m \\
v(t, 0, m, z) &= \psi(t, m, z) \\
v_m(t, m, m, z) &= 0 \\
v(T, a, m, z) &= a,
\end{align*}
\]

where

\[
H(a, m, z, v, v_a, v_z, v_{aa}) = \\
\sup_{0 \leq w \leq \beta m} \left\{ f(w, m) + w(v_z - v_a) \right\} - rv + (r - q)a v_a + \frac{1}{2} \sigma^2 a^2 v_{aa}
\]

\[
H_0(a, v, v_a, v_{aa}) = -rv + (r - q)a v_a + \frac{1}{2} \sigma^2 a^2 v_{aa}
\]

\[
\psi(t, m, z) = \sup_{0 \leq w \leq \beta m} \int_t^T e^{-r(s-t)} f(w_s, m) ds \text{ such that } \int_t^T w_s ds \leq \gamma m - z.
\]

Non-linear parabolic PDE on \([0, T] \times \mathbb{R}^3\) with a free boundary and unusual boundary conditions
Reducing the dimension

**Theorem**

\[ V(t, a, m, z) = mW(t, a/m, z/m) = mW(t, x, y), \] where \( W \) is a viscosity solution of

\[
\begin{align*}
\min(-v_t - G(x, y, v, v_x, v_y, v_{xx}), v - (1 - k)x) &= 0 \text{ for } (x, y) \in (0, 1) \times [0, \gamma) \\
\min(-v_t - G_0(x, v, v_x, v_{xx}), v - (1 - k)x) &= 0 \text{ for } (x, y) \in (0, 1) \times \{\gamma\} \\
v(t, 0, y) &= \zeta(t, y) \\
v_x(t, 1, y) + yv_y(t, 1, y) &= v(t, 1, y) \\
v(T, x, y) &= x,
\end{align*}
\]

where

\[
G(x, y, v, v_x, v_y, v_{xx}) = \sup_{0 \leq u \leq \beta} \{u(v_y - v_x) + f(u, 1)\} - rv + (r - q)xv_x + \frac{1}{2} \sigma^2 x^2 v_{xx}
\]

\[
G_0(x, v, v_x, v_{xx}) = -rv + (r - q)xv_x + \frac{1}{2} \sigma^2 x^2 v_{xx}
\]

\[
\zeta(t, y) = \sup_{0 \leq u \leq \beta} \int_t^T e^{-r(s-t)} f(u_s, 1) ds \text{ such that } \int_t^T u_s ds \leq \gamma - y.
\]

Non-linear parabolic PDE on \([0, T] \times [0, 1] \times [0, \gamma]\) with a free boundary and (even more) unusual boundary conditions
Worst case strategy

The worst case withdrawal strategy \( \hat{w}(t, A_t, M_t, Z_t) \) is given by the maximizer of

\[
w \mapsto f(w, M_t) + w \left[ V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t) \right]
\]
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- The worst case surrender time is

$$\theta = \inf \{t \geq 0 : V(t, A_t, M_t, Z_t) = (1 - k(t))A_t\}$$
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  \]

- The issuer can set \( k(t) \) so that the worst case policy holder never surrenders early
Numerical scheme

- Semi-Lagrangian scheme with an obstacle
- Backwards in time
- Solves an optimization problem in every time-step
- Converges to the true solution if the mesh size of the discretization goes to zero
- Gives approximations to $V(t, a, m, z)$ and the worst case behavior $\hat{w}$ and $\hat{\theta}$
Hedging

The issuer can super-hedge the contract by trading in $S_t$ and the money market account.
Hedging

The issuer can super-hedge the contract by trading in $S_t$ and the money market account.

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
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- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
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Proof: Itô's formula
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- Hold the amount $1_{\{A_t > 0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
Hedging

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Hedging

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- Keep the rest of the portfolio value in the money market account
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- This will super-hedge the contract
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- It will exactly hedge the contract if $(w, \theta)$ equals the worst case strategy $(\hat{w}, \hat{\theta})$
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- It will exactly hedge the contract if $(w, \theta)$ equals the worst case strategy $(\hat{w}, \hat{\theta})$

Proof: Itô’s formula
Numerical results

The graph shows the risk neutral price of a VA+GMWB contract as a function of time to maturity for different values of the interest rate $r$ and volatility $\sigma$. The price is increasing in $\sigma$ and decreasing in $r$, which makes these products difficult to sell in a low interest rate environment.
Numerical results

<table>
<thead>
<tr>
<th>Time to Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>1.05</td>
</tr>
<tr>
<td>4</td>
<td>1.10</td>
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<tr>
<td>6</td>
<td>1.15</td>
</tr>
<tr>
<td>8</td>
<td>1.20</td>
</tr>
<tr>
<td>10</td>
<td>1.25</td>
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<td>12</td>
<td>1.30</td>
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<tr>
<td>14</td>
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<tr>
<td>16</td>
<td>1.40</td>
</tr>
<tr>
<td>18</td>
<td>1.45</td>
</tr>
<tr>
<td>20</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Risk Neutral Price of VA+GMWB Contract

- Price is increasing in $\sigma$
- Price is decreasing in $r$

...not taken into account by insurance companies
...makes these products difficult to sell in a low interest rate environment
Price is increasing in $\sigma$ ... not taken into account by insurance companies
Numerical results

- Price is increasing in $\sigma$ ... not taken into account by insurance companies
- Price is decreasing in $r$
Numerical results

- Price is increasing in $\sigma$ ... not taken into account by insurance companies
- Price is decreasing in $r$ ... makes these products difficult to sell in a low interest rate environment
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$

- Price is increasing in $x = a/m$ and decreasing in $y = z/m$
Price and hedging ratio as functions of $x = \frac{a}{m}$ and $y = \frac{z}{m}$

- Price is increasing in $x = \frac{a}{m}$ and decreasing in $y = \frac{z}{m}$
- The hedge is always long in $S_t$ in contrast to the hedge of a put option)
Worst case withdrawal and surrender

- $T - t = 5$ years, $r = 1\%$, $\sigma = 18\%$, $q = 0.8\%$, $p_1 = 0$, $p_2 = 5\%$, $k \equiv 0$
- yellow = full withdrawal (at rate $\beta M_t$)
- light brown = intermediate withdrawal (at rate $\alpha M_t$)
- brown = no withdrawal
- black = surrender
Discouraging early surrender

Set the surrender penalty function such that

\[ k(t) \geq (T - t)q \]
<table>
<thead>
<tr>
<th>Introduction</th>
<th>The Model</th>
<th>The HJB Equation</th>
<th>Numerical Scheme</th>
<th>Hedging</th>
<th>Numerical Results</th>
</tr>
</thead>
</table>

Merci!