Capital allocation beyond Euler

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- Capital allocation for portfolios

- Capital allocation on risk factors

- Case study
Why capital allocation?

- “Just” calculating solvency capital is not enough!
  - Capital requirement needs to be understood and integrated into business and strategy.

- Capital allocation splits the total required/target capital $C$ into amounts $C_1, \ldots, C_n$ with

$$C = \sum_{i=1}^{n} C_i$$

where each $C_i$ is an amount of capital related to a risk factor or part of the business.

- Capital allocation is a tool to answer important questions about your business:
  - What are your greatest risks?
  - What are the sources of diversification?
  - Are you adequately rewarded for the risks you take?
  - How can you optimise risk-return?

- Under Solvency II it is required as part of the use test and the ORSA
Capital allocation for portfolios of risk

- The capital allocation for a portfolio of risks is the most important special case of allocation.
- Portfolio of risks means the total P&L or loss function is a sum:

\[
\text{TOT} = \sum_{i=1}^{n} X_i
\]

  - TOT: Total P&L or total loss
  - \( X_i \): P&L or Loss of portfolio components, risk factors

- **Euler method**: Method to allocate capital \( C_i \) to the components \( X_i \) of a portfolio of risks
  - Has very nice properties
  - Easy to calculate (for many risk measures)
  - Intuitive interpretation (for many risk measures)

- There are many examples of portfolio of risks where the Euler method is used in practice
  - Allocation to financial instruments in an investment portfolio
  - Allocation to insurance contracts in an insurance portfolio
  - Allocation to lines of business
  - Allocation to legal entities of a group
Example: Expected Shortfall

- The risk measure Expected Shortfall allows a particularly nice Euler allocation.
- Expected Shortfall is estimated as average of worst outcomes of a simulation. In the figure at 10% level: $C = -E[TOT \mid TOT < q_{10\%}]$

Simulated portfolio P&L

Sorted portfolio P&L

Sort the sample

Tail: 10% worst scenarios

Average = -44
Capital = 44
Example: Joint simulation

- Example: A portfolio of three risks with \( \text{TOT} = X_1 + X_2 + X_3 \)
  - Joint simulation with \( N = 1000 \) of the P&L of the four variables.
  - Each row is an independent sample.
  - Each column a variable.
Example: Sorted outcomes

- Sort rows according to TOT the total P&L: Good outcomes of TOT on top bad ones at the bottom.
  - X3 and (to a lesser extent) also X2 are bad if TOT is bad.
  - X1 seems to be undetermined.
Example: Allocation of Expected Shortfall

The Euler allocation for X1, X2 and X3 is their tail average according to the sort order of TOT. Total capital: $C = 44$ allocated capital: $C_1 = -8$ $C_2 = 12$ $C_3 = 40$

Euler allocation always sums up to total capital!

$$C = C_1 + C_2 + C_3$$

$$E[TOT \mid TOT < q_{10\%}] = E[X_1 \mid TOT < q_{10\%}] + E[X_2 \mid TOT < q_{10\%}] + E[X_3 \mid TOT < q_{10\%}]$$
Euler allocation as a useful tool

- The Euler allocations has nice properties:
  - Allocated capital sums up to total capital
  - Allocation can be computed from simulations
  - Intuitive interpretation

- Euler is the only method which provides all the answers:
  - Largest risk? ⇒ Risk factor with largest allocated capital
  - Diversification? ⇒ Allocated capital smaller than stand-alone capital
  - Measure reward? ⇒ Return On Risk Adjusted Capital (RORAC)
    Expected return (total or component) divided by (total or allocated) capital.
  - Optimisation? ⇒ RORAC compatibility: Increasing exposure to component with
    largest component-RORAC will increase RORAC of total portfolio
- Capital allocation for portfolios

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BUT: Not all risks come as a portfolio!

- Portfolios of risks are common but there are many examples where risk factors combine in a non-linear fashion.
- Discounted or FX cash flows \( f(X, Y) = X \cdot Y \)
  - \( X \) (insurance) cash flow
  - \( Y \) discount factor or FX rate
- Excess of loss treaty with multiple perils
  \[ f(X, Y) = \max(X + Y - c, 0) \]
  - \( X, Y \) perils e.g. earthquake, hurricane
  - \( c \) deductible
- Example: Financial return guarantee on a mixed investment portfolio
  \[ f(X, Y) = \max(X + Y - c, 0) \]
  - \( X, Y \) asset classes, \( c \) guarantee/strike level
- How does capital allocation actually work in those cases?
- In these cases there is currently no “gold-standard” for allocation comparable to Euler allocation.
What is the problem?

- Immediately obvious algebraic problem:
  \[ E[TOT \mid TOT < q_{10\%}] = E[X_1 \mid TOT < q_{10\%}] + E[X_2 \mid TOT < q_{10\%}] + E[X_3 \mid TOT < q_{10\%}] \]
  works only for \( TOT = X_1 + X_2 + X_3 \).

- Deeper conceptual problem:
  - The marginal principle \( C[X_i] = C[TOT] - C[TOT - X_i] \) breaks down because \( TOT - X_i \) has no meaning for non-additive risk factors.
  - Euler principle is infinitesimal version of the marginal principle

- From a business perspective:
  - Euler allocation is closely related to what you can actually DO with a portfolio: Increase/Decrease the exposures to the single risk factors.
  - When discounting a cash-flow you can’t increase/decrease the exposure to the discount factor.
  - If you can’t change the exposure RORAC compatibility is pretty useless
What can be done?

- Loss allocation according to the Cat model vendors: Allocate loss in a simulation year to the risk factor (event) which causes the bond/insurance contract to trigger.
  - Works only for event type risk factors
  - Ignores interaction of events (for example: Aggregate covers)
  - Has poor statistical qualities

- Split by risk category
  - Capital per risk category is routinely reported.
  - But risk factors such as interest (or FX) rates enter into all lines of business and investments. How are they carved out from the rest?
  - What does “diversification” mean?
  - Can this serve as a basis for capital allocation?

<table>
<thead>
<tr>
<th>Risk Category</th>
<th>Capital (in F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insurance (P&amp;C)</td>
<td>2.0</td>
</tr>
<tr>
<td>Insurance (L&amp;H)</td>
<td>1.4</td>
</tr>
<tr>
<td>Market</td>
<td>2.1</td>
</tr>
<tr>
<td>Credit</td>
<td>0.9</td>
</tr>
<tr>
<td>Operational</td>
<td>0.7</td>
</tr>
<tr>
<td>Required capital w/o diversification</td>
<td>7.1</td>
</tr>
<tr>
<td>Diversification</td>
<td>4.4</td>
</tr>
<tr>
<td>Required capital</td>
<td>-2.7</td>
</tr>
</tbody>
</table>

**Generic example of a split by risk category**
Split by “Freezing-the-Margins”

- Split by freezing the margins might be the most popular method to calculate capital per risk factor. Example: Split capital for a P&L model $f(X,Y)$ with risk factors insurance risk ($X$) and market risk ($Y$) into capital for insurance and market risk.

- **Step 1:** Define “pure insurance risk” by replacing all stochastic inputs $Y$ for market risk with a constant value $y_0$: $INS(X) = f(X, y_0)$

- **Step 2:** Define “pure market risk” by replacing $X$ with the constant value $x_0$: $MKT(Y) = f(x_0, Y)$

- **Step 3:** Run the model three times to calculate the “stand-alone” capitals for $INS$ and $MKT$ and the total risk $TOT$.
  - Capital for insurance risk $C_{INS} = C[INS(X)] = C[f(X, y_0)]$
  - Capital for market risk $C_{MKT} = C[MKT(Y)] = C[f(x_0, Y)]$
  - Total capital $C = C_{TOT} = C[f(X, Y)]$

- **Step 4:** Add up and call the difference “diversification”

$$C_{TOT} = C_{INS} + C_{MKT} - \text{Diversification}$$

Split by freezing-the-margins seems to be quite intuitive but has three problems!
The problems with freezing-the-margins

- **First problem:** The “pure” models do not add up!
  \[ f(X,Y) \neq f(x_0,Y) + f(X,y_0) \]

- **Solution:** A residual term needs to be included in the allocation
  \[ f(X,Y) = f(x_0,Y) + f(X,y_0) + RES \]
  Split of \( C_{TOT} \) into \( C_{INS}, C_{MKT}, C_{RES} \)

- **Second problem:** The allocated capitals do not add up to the total capital.
- **Solution:** Use Euler allocation instead of stand-alone capital.

- **Third problem:** What do the terms \( INS(X) = f(X, y_0) \) and \( MKT(Y) = f(x_0, Y) \) represent in terms of business or in terms of modelling?
  - The terms have no consistent interpretation in terms of business
  - Lack of interpretation makes the choice of constants \( x_0, y_0 \) and the capital split arbitrary.
  - Simply replacing a random variable with a constant is not a consistent stochastic approach
A general framework

- Step 1: Split the total into a sum of components each depending on one single risk factor only – the “pure risk” functions – and the residual.

\[ f(X, Y) = INS(X) + MKT(Y) + RES(X, Y) \]

- Step 2: Use Euler allocation to allocate capital onto each component.

\[ f(X, Y) = INS(X) + MKT(Y) + RES(X, Y) \]

Euler allocation

\[ C = C_{INS} + C_{MKT} + C_{RES} \]

- The hard problem is the split into a sum, i.e. Step 1!

- The split should be based on principles
  - **Principle 1:** A split should be based on real world business considerations
  - **Principle 2:** A split should be mathematically sound and consistent
Split by optimal hedging

- The mathematical idea of split by optimal hedging is: Approximation.
  - Choose the pure models such that the residual term $RES$ is as small as possible:
    
    \[
    \text{Find } h \text{ and } g \text{ such that } f(X,Y) - h(X) - g(Y) \rightarrow \text{minimal}
    \]

- The business idea behind split by optimal hedging is .... optimal hedging (or optimal reinsurance).
  - $MKT(Y)$, the optimal $g(Y)$, is the best hedge of the total P&L $f(X,Y)$ using only market risk instruments.
  - $INS(X)$, the optimal $h(X)$, is the best reinsurance of the total P&L $f(X,Y)$ using only reinsurance contracts not mentioning market risk.
  - $RES(X,Y)$ is the remaining basis risk.
Concrete implementation: Variance hedging

- Some specifications are required to turn split by hedging into a practical approach
  - What is the universe of permitted hedges or reinsurance contracts?
  - What is the metric to determine “optimal”?
  - How can these be calculated in practice?

- Metric: minimal variance (least squares)
  - Optimal solutions are conditional expectations, i.e. the mathematics is sound and well understood.

- Permitted instruments/pure models
  - Choice depends on $f$ and practical considerations
  - Typically parametric families (see next section)

- Practical calculations
  - Least squares is easy using regression techniques
  - Big advantage: Just a single model run required no matter how many risk factors there are in the split.
Does the method make a difference?

- It is not difficult to test typical functions over a range of relevant distributional assumptions and compare the results of the various splitting methods.

- Some observations for $f(X, Y) = X \cdot Y$
  - The residual term in the split freeze can be substantial (>20% of total capital) especially for correlated risk factors.
  - For independent risks split freeze and variance hedging are exactly identical.
  - For correlated risks they are different, differences can be 10% of total capital or more.
  - One of the causes of differences is cross-hedging of correlated risk, which is ignored by the freeze approach.

- Some observations for $f(X, Y) = \max(X + Y - c, 0)$
  - Behaviour for the freeze method depends strongly on interplay between deductible $c$ and the frozen points $x_0, y_0$.
  - For low deductibles $f$ is like $X + Y$ and freeze and variance methods produce similar results.
  - For higher deductibles residual terms can get very large.
  - Freeze for higher deductibles seems quite erratic (allocating 0% or 100%).
  - Differences between methods for high deductibles are huge.
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- Case study
The cat bond index

- This case study is joint work with Jiven Gill from Schroders investment!

- Swiss Re Global Cat bond index:
  - A portfolio of cat bonds designed to reflect the returns of the catastrophe bond market
  - Swiss Re Capital Markets launched the Index in 2007
  - First total return index for the sector.

- The question: “What are the largest risks contributing to losses for the Swiss Re Cat Bond index?”
Cat bond pay-out is non-linear

- Pay-out profile of a Cat bond on some kind of loss from natural catastrophes
The challenge: Cat bonds are not “pure risk”

- Cat Bond payoffs can depend on more than one type of natural disaster (peril)
  - Return $f(x,y,z)$ might depend on $x$: California earthquake losses, $y$: Florida Hurricane losses, $z$: European windstorm losses
  - Depending on the functional form $f(.)$, cat bond can be triggered due to losses from only one of the perils or from a combination of them.
  - Over 40% of the cat bonds in the Swiss Re Index are multi-peril bonds.

- The answer in four steps:
  - Step 1: Find “pure risk” functions to describe cat bonds returns
  - Step 2: Split each individual cat bond into a sum of “pure risk” functions
  - Step 3: Define the cat bond index as the weighted sum of the individual cat bonds “pure risk” functions
  - Step 4: Use Euler allocation of Expected Shortfall
Definition of the pure risk functions

- Parametric families of simple single peril instruments (“calls”) are the building blocks of the pure risk functions:
  \[ g_i(X) = \max(X - c_i, 0) \]
  \( X \): denotes industry losses due a single peril such as industry loss from Florida Tropical Cyclone
  \( c_i \): deductible or attachment level of instrument \( i \)

- The pure risk functions are constructed from linear combinations fitted by ordinary least squares
  \[ d_X(X) = \sum_{i=1}^{n} \beta_i \cdot \max(X - c_i, 0) \]

- There are pure risk functions for all perils/regions to replicate all bonds
  \[ f(X, Y, Z, \ldots) = d_X(X) + d_Y(Y) + d_Z(Z) + \cdots + RES(X, Y, Z, \ldots) \]

- Industry losses per perils and regions for calibration were extracted from AIR Catrader®
Allocation of Expected Shortfall

- A model of “pure” risk functions which adds up to 100%
- Each individual risk factor in the model has a business and economical meaning.

1% Expected Shortfall contribution (in %)
Cat Bond index as sum of pure risk functions

- The decomposition allows analysis beyond loss allocation

\[ R_{\text{Cat Bond index}} = R_{\text{Florida}_{-}TC} + R_{\text{California}_{-}EQ} + \ldots + \text{RES} \]

Red points are the pure risk functions
Overall fit is reasonably well even though there are two sources of error:

- Errors due to the payoff function: \( f(x, y) \neq f_1(x) + f_2(y) \)
- Errors due to risk factors: The pure risk instruments are based on *industry losses*, while bonds might insure company specific portfolios or have parametric triggers.

Scatterplot of portfolio returns

Exceedance Probability Curves
Further reading

- Find below some papers on the topic. But be warned: The literature is (still) quite technical!

  - "Decomposing life insurance liabilities into risk factors" (2015)
    Schilling, K., Bauer, D., Christiansen, M., Kling, A.,
    [Link](https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi2/dokumente/preprint-server/2016/2016-03.pdf)

  - “Risk Capital Allocation and Risk Quantification in Insurance Companies” (2012)
    Ugur Karabey, [Link](http://hdl.handle.net/10399/2566)

  - “Risk factor contributions in portfolio credit risk models” (2010)
    Dan Rosen, David Saunders,
    [Link](https://www.researchgate.net/publication/222695088_Risk_factor_contributions_in_portfolio_credit_risk_models)

  - “Capital Allocation to Business Units and Sub-Portfolios: the Euler Principle” (2008)
    Dirk Tasche, [Link](https://arxiv.org/abs/0708.2542)

  - “Relative importance of risk sources in insurance systems” (1998)
    Edward Frees, [Link](http://dx.doi.org/10.1080/10920277.1998.10595694)
Contact details

- If you know of other ways to split or – even better – a new way to allocate, let me know!

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