

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each problem on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten sides of summary are allowed

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Last name	
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Student number (if available)	

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Question	Points	Control	Maximum
#1			12
#2			10
#3			10
#4			10
#5			8
Total			50

Question 1 (12 Pts)

- a) Let X be a random variable with a standard Laplace distribution; that is, the cdf of X is

$$F(x) = \begin{cases} \frac{1}{2} \exp(x) & \text{if } x \leq 0 \\ 1 - \frac{1}{2} \exp(-x) & \text{if } x \geq 0. \end{cases}$$

Calculate $\text{VaR}_\alpha(X)$ and $\text{AVaR}_\alpha(X)$ for $\alpha \in [1/2, 1)$. (3 Pts)

- b) Let X be a random variable such that $\mathbb{E}[|X|] < \infty$. Show that

$$\text{AVaR}_\alpha(X) = \text{VaR}_\alpha(X) + \frac{1}{1-\alpha} \mathbb{E}[(X - \text{VaR}_\alpha(X))_+]$$

for all $\alpha \in (0, 1)$. (3 Pts)

- c) Name one advantage of VaR over AVaR and one advantage of AVaR over VaR. (2 Pts)

- d) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider the risk measure $\rho: L^1(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ given by

$$\rho(X) = \max\{\text{AVaR}_{0.75}(X), \text{VaR}_{0.95}(X)\}.$$

Which properties of a coherent risk measure does ρ have? Please, justify your answers. (4 Pts)

Solution 1

- a) $\text{VaR}_\alpha(X)$, $\alpha \geq 1/2$, can be obtained by inverting $F|_{\mathbb{R}_+}$. This gives

$$\text{VaR}_\alpha(X) = -\log(2(1-\alpha))$$

for all $\alpha \geq \frac{1}{2}$. Hence, using a simple change of variable we obtain

$$\begin{aligned} \text{AVaR}_\alpha(X) &= \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(X) \, du \\ &= -\frac{1}{1-\alpha} \int_\alpha^1 \log(2(1-u)) \, du \\ &= -\frac{1}{2(1-\alpha)} \int_0^{2(1-\alpha)} \log(s) \, ds \end{aligned}$$

for all $\alpha \in [1/2, 1)$. Since $\lim_{x \rightarrow 0} x \log(x) = 0$ we obtain

$$\begin{aligned} \text{AVaR}_\alpha(X) &= -\frac{1}{2(1-\alpha)} \int_0^{2(1-\alpha)} \log(s) \, ds \\ &= -\frac{1}{2(1-\alpha)} (2(1-\alpha) \log(2(1-\alpha)) - 2(1-\alpha)) = 1 - \log(2(1-\alpha)) \end{aligned}$$

for all $\alpha \in [1/2, 1)$.

b) The stated identity follows by direct computation. In fact, we have

$$\begin{aligned}
\text{AVaR}_\alpha(X) &= \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(X) du \\
&= \text{VaR}_\alpha(X) + \frac{1}{1-\alpha} \int_\alpha^1 (\text{VaR}_u(X) - \text{VaR}_\alpha(X)) du \\
&= \text{VaR}_\alpha(X) + \frac{1}{1-\alpha} \int_0^1 (\text{VaR}_u(X) - \text{VaR}_\alpha(X)) \mathbb{1}_{(\alpha,1)}(u) du \\
&= \text{VaR}_\alpha(X) + \frac{1}{1-\alpha} \mathbb{E}_U \left[(q_U^-(X) - \text{VaR}_\alpha(X))_+ \right]
\end{aligned}$$

where $U \sim \text{Unif}(0,1)$, $\alpha \in (0,1)$. Using the quantile transformation theorem, we obtain

$$\text{AVaR}_\alpha(X) = \text{VaR}_\alpha(X) + \frac{1}{1-\alpha} \mathbb{E}[(X - \text{VaR}_\alpha(X))_+]$$

for all $\alpha \in (0,1)$.

- c)
- VaR is defined on $L^0(\mathbb{P})$ but AVaR only on $L^1(\mathbb{P})$.
 - VaR is in general not subadditive as an example showed in the lecture, but due to a representation theorem proved in class AVaR is a coherent risk measure.
 - VaR is a frequency measure, i.e. it does not see what happens in the tails, whereas AVaR is a severity measure and therefore incorporates the behaviour in the tails.
 - AVaR is much more difficult to estimate and backtest than VaR.
- d) First recall that a risk measure is called coherent if it satisfies the axioms of monotonicity (M), positive homogeneity (P), subadditivity (S) and has the translation property (T). Moreover, we have seen in the lecture that VaR satisfies (M), (T), (P) but in general not (S) and that AVaR is a coherent risk measure. Since the maximum has the following properties

1. $\max\{x+a, y+a\} = \max\{x, y\} + a$ for all $x, y, a \in \mathbb{R}$;
2. $\max\{\lambda x, \lambda y\} = \lambda \max\{x, y\}$ for all $x, y \in \mathbb{R}$ and all $\lambda > 0$;
3. $\max\{x+y, z+w\} \leq \max\{x, z\} + \max\{y, w\}$ for all $x, y, z, w \in \mathbb{R}$.

ρ satisfies (M), (P) and (T), but (S) does not hold. To see this it suffices to find a pair of random variable (X_1, X_2) such that

$$\rho(X_i) = \text{AVaR}_{0.75}(X_i) \quad \text{and} \quad \text{VaR}_{0.95}(X_1 + X_2) > \text{AVaR}_{0.75}(X_1) + \text{AVaR}_{0.75}(X_2).$$

In fact, in this case we obtain

$$\rho(X_1 + X_2) \geq \text{VaR}_\beta(X_1 + X_2) > \text{AVaR}_\alpha(X_1) + \text{AVaR}_\alpha(X_2) = \rho(X_1) + \rho(X_2).$$

Next we give a simple example of this failure. Pick two independent $\text{Ber}(p)$ -distributed random variables X_1, X_2 with success probability $p \in (0,1)$; that is, their common cdf is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ 1-p, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

Since $\text{VaR}_\alpha(X_i) = \inf\{x: F(x) \geq \alpha\}$, $i = 1, 2$, we obtain

$$\text{VaR}_\alpha(X_i) = \begin{cases} 0, & 0 < \alpha \leq 1 - p, \\ 1, & 1 - p < \alpha < 1. \end{cases}$$

and

$$\text{AVaR}_\alpha(X_i) = \begin{cases} \frac{p}{1-\alpha}, & \text{if } \alpha \in (0, 1 - p] \\ 1, & \text{if } \alpha \in (1 - p, 1) \end{cases}$$

for $i = 1, 2$. On the other hand, the independence of X_1 and X_2 shows

$$X_1 + X_2 = \begin{cases} 0 & \text{with } \mathbb{P}[X_1 + X_2 = 0] = (1 - p)^2, \\ 1 & \text{with } \mathbb{P}[X_1 + X_2 = 1] = 2p(1 - p), \\ 2 & \text{with } \mathbb{P}[X_1 + X_2 = 2] = p^2 \end{cases}$$

and therefore

$$F_{X_1+X_2}(x) = \begin{cases} 0, & x < 0, \\ (1 - p)^2, & 0 \leq x < 1, \\ 1 - p^2, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

As above, this shows

$$\text{VaR}_\alpha(X_1 + X_2) = \begin{cases} 0, & 0 < \alpha \leq (1 - p)^2, \\ 1, & (1 - p)^2 < \alpha \leq 1 - p^2, \\ 2, & 1 - p^2 < \alpha < 1. \end{cases}$$

If we choose $p = 0.04$, then we have $0.95 < 1 - p = 0.96$ and $(1 - p)^2 = 0.9216 < 0.95$, from which we obtain $\text{VaR}_{0.95}(X_i) = 0$, and therefore,

$$\rho(X_1) + \rho(X_2) = 2 \text{AVaR}_{0.75}(X_1) = 2 \frac{4}{25} < 1 = \text{VaR}_{0.95}(X_1 + X_2) \leq \rho(X_1 + X_2).$$

Question 2 (10 Pts)

- a) Let $X_i \sim S_d(\psi_i)$, $i = 1, \dots, n$, be independent random vectors and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Show that $Z = \sum_{i=1}^n \alpha_i X_i$ is spherically distributed. (3 Pts)
- b) Assume that the daily losses of an investment during the next t days are given by

$$(X_1, \dots, X_t) \sim M_t(0, \Sigma, \hat{F}_W)$$

for a non-negative random variable W and a $t \times t$ -matrix $\Sigma = \sigma^2 P$, where $\sigma > 0$ is a constant and P a correlation matrix with $P_{ij} = \rho$ for all $i \neq j$. Show that there exists a function $f: \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\text{VaR}_\alpha(X_1 + \dots + X_t) = f(t) \text{VaR}_\alpha(X_1)$$

for all $\alpha \in (0, 1)$. Can you compute f explicitly? (4 Pts)

- c) Let $X \sim E_d(\mu, \Sigma, \psi)$ and $Y \sim E_d(\nu, \Sigma, \varphi)$ be two independent random vectors. Is $Z = X + Y$ again elliptically distributed? If yes, derive $m \in \mathbb{R}^d$, $M \in \mathbb{R}^{d \times d}$ and $\xi: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $Z \sim E_d(m, M, \xi)$. If no, give a counterexample. (3 Pts)

Solution 2

- a) First recall that a random variable Z is spherical if $UZ \stackrel{(d)}{=} Z$ for every orthogonal matrix $U \in \mathbb{R}^{d \times d}$. Thus it suffices to prove $\varphi_Z = \varphi_{UZ}$ for every orthogonal matrix $U \in \mathbb{R}^{d \times d}$ where φ_Z, φ_{UZ} stand for the characteristic function of Z, UZ , respectively. Thus using that $X_j \sim S_d(\psi_j), 1 \leq j \leq n$, and they are independent we obtain

$$\begin{aligned}\varphi_{UZ}(t) &= \mathbb{E}[e^{it \cdot UZ}] = \mathbb{E}[e^{i(U^T t) \cdot Z}] = \mathbb{E}[e^{i \sum_{j=1}^n (\alpha_j U^T t) \cdot X_j}] = \prod_{j=1}^n \mathbb{E}[e^{i(\alpha_j U^T t) \cdot X_j}] \\ &= \prod_{j=1}^n \psi_j(\|\alpha_j U^T t\|^2) = \prod_{j=1}^n \psi_j(\|\alpha_j t\|^2) = \prod_{j=1}^n \varphi_{X_j}(\alpha_j t) = \varphi_Z(t)\end{aligned}$$

for every orthogonal matrix $U \in \mathbb{R}^{d \times d}$ and all $t \in \mathbb{R}^d$, which in turn shows that Z is spherically distributed.

- b) As seen in the lecture, if $X \sim M_t(0, \Sigma, \widehat{F}_W)$ then we have

$$X_1 + \dots + X_t = 1^\top X \sim M_1(0, 1^\top \Sigma 1, \widehat{F}_W)$$

where $1 = (1, \dots, 1) \in \mathbb{R}^t$. Moreover, its scale matrix can be rewritten as

$$1^\top \Sigma 1 = \sigma^2 1^\top P 1 = \sigma^2 \left(\sum_{i=1}^t 1 + \sum_{\substack{i,j=1 \\ i \neq j}}^t \rho \right) = \sigma^2 (t + t(t-1)\rho) = t\sigma^2 (1 + \rho(t-1)).$$

On the other hand, $1^\top X$ admits a stochastic representation as

$$1^\top X \stackrel{(d)}{=} \sqrt{1^\top \Sigma 1} \sqrt{W} Z = \sqrt{t(1 + \rho(t-1))} \sigma \sqrt{W} Z \stackrel{(d)}{=} \sqrt{t(1 + \rho(t-1))} X_1,$$

where $Z \sim N(0, 1)$. Since VaR is a positive homogeneous, distribution-based risk measure, we get

$$\text{VaR}_\alpha(X_1 + \dots + X_t) = \text{VaR}_\alpha\left(\sqrt{t(1 + \rho(t-1))} X_1\right) = \sqrt{t(1 + \rho(t-1))} \text{VaR}_\alpha(X_1)$$

for all $\alpha \in (0, 1)$.

- c) Yes, the random variable Z is again elliptically distributed. To see this, let us derive the characteristic function ϕ_Z of Z . It follows from the independence of X and Y that $\phi_Z(u) = \phi_X(u) \phi_Y(u)$ for all $u \in \mathbb{R}^d$. Using the formula for the characteristic function of an elliptical distribution derived in class we obtain

$$\phi_Z(u) = \phi_X(u) \phi_Y(u) = e^{iu^\top \mu} \psi(u^\top \Sigma u) e^{iu^\top \nu} \varphi(u^\top \Sigma u) = e^{iu^\top (\mu + \nu)} \psi(u^\top \Sigma u) \varphi(u^\top \Sigma u)$$

for all $u \in \mathbb{R}^d$. Defining $m = \mu + \nu$, $M = \Sigma$ and $\xi(u) = \psi(u) \varphi(u)$, we further get

$$\phi_Z(u) = e^{iu^\top m} \xi(u^\top M u), \quad u \in \mathbb{R}^d.$$

It thus follows that $Z \sim E_d(m, M, \xi)$.

Question 3 (10 Pts)

- a) Let X be an $\text{Exp}(\lambda)$ -distributed random variable for a parameter $\lambda > 0$. Calculate the distribution function and the moments of $Y = \exp(X)$. (3 Pts)
- b) Does Y have a density? If yes, can you compute it? (1 Pts)
- c) Now, consider a two-dimensional random vector (X_1, X_2) such that $X_i \sim \text{Exp}(\lambda_i)$ for parameters $\lambda_i > 0, i = 1, 2$. Under which conditions does the linear correlation between $Y_1 = \exp(X_1)$ and $Y_2 = \exp(X_2)$ exist? (2 Pts)
- d) Assume $\lambda_1 = 3$ and $\lambda_2 = 4$. What is the range of possible correlations between Y_1 and Y_2 ? (4 Pts)

Solution 3

- a) The pdf of $X \sim \text{Exp}(\lambda)$ is given by $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Thus, the distribution of X is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

for all $x > 0$ and $F_X(x) = 0$ for all $x \leq 0$. This in turn shows

$$F_Y(y) = \mathbb{P}[\exp(X) \leq y] = F_X(\log(y)) = 1 - y^{-\lambda}$$

for all $y > 1$ and $F_Y(y) = 0$ for all $y \leq 1$. A straight forward calculation shows

$$\mathbb{E}[Y^k] = \lambda \int_0^\infty e^{kz} e^{-\lambda z} dz = \frac{\lambda}{k - \lambda} e^{(k-\lambda)z} \Big|_0^\infty = \frac{\lambda}{\lambda - k}$$

for all $k \in \mathbb{N}$ such that $k < \lambda$ and otherwise $\mathbb{E}[Y^k] = \infty$

- b) Since the cdf F_Y from a) is smooth on $(1, \infty)$ its pdf is given by

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{\lambda}{y^{\lambda+1}}$$

for all $y > 1$ and otherwise vanishes.

- c) The linear correlation of X_1, X_2 exists if $X_i \in L^2(\mathbb{P})$ and $\text{Var}(X_i) > 0$ for $i = 1, 2$. Using b) we conclude that this is equivalent to $\min\{\lambda_1, \lambda_2\} > 2$ as in this case the second condition is automatically satisfied.
- d) Since $\min\{\lambda_1, \lambda_2\} = 3$, exercise c) shows that the linear correlation is well-defined. Hence, Hoeffding's identity implies $\rho \in [\rho_{\min}, \rho_{\max}]$ whereas the the minimal, maximal linear correlation is attained if Y_1 and Y_2 are coupled by the counter-monotonicity, comonotonicity copula $W(u, v) = (u + v - 1)_+$, $M(u, v) = \min\{u, v\}$, respectively. Hence, we have $\rho_{\min} = \rho(Y_1, Y_2)$ and $\rho_{\max} = \rho(Y_1, Y_2)$ in the respective cases. To calculate ρ_{\min}, ρ_{\max} explicitly, we need $\mathbb{E}[Y_1], \mathbb{E}[Y_2], \text{Var}(Y_1), \text{Var}(Y_2)$ and $\text{Cov}(Y_1, Y_2)$. Using the calculations from b), one easily obtains

$$\mathbb{E}[Y_1] = \frac{3}{2}, \mathbb{E}[Y_2] = \frac{4}{3}, \text{Var}(Y_1) = \frac{3}{4}, \text{Var}(Y_2) = \frac{2}{9}.$$

Finally, to compute $\text{Cov}(Y_1, Y_2)$ we need $\mathbb{E}[Y_1 Y_2]$. If Y_1, Y_2 are coupled by the counter-monotonicity, comonotonicity copula, respectively, we know $(Y_1, Y_2) \stackrel{(d)}{=} (q_{Y_1}(U), q_{Y_2}(1-U))$, $(Y_1, Y_2) \stackrel{(d)}{=} (q_{Y_1}(U), q_{Y_2}(U))$ for some $U \sim \text{Unif}(0, 1)$ and q_{Y_1}, q_{Y_2} are quantile functions of Y_1, Y_2 . By inverting the distribution functions of Y_1, Y_2 we obtain

$$q_{Y_1}(u) = (1-u)^{-1/3} \quad q_{Y_2}(v) = (1-v)^{-1/4}$$

for all $u, v \in (0, 1)$. Hence, we obtain

$$\mathbb{E}[Y_1 Y_2] = \mathbb{E}[h(Y_1, Y_2)] = \mathbb{E}[h(q_{Y_1}(U), q_{Y_2}(1-U))] = \int_0^1 (1-x)^{-1/3} x^{-1/4} dx$$

and

$$\mathbb{E}[Y_1 Y_2] = \mathbb{E}[h(Y_1, Y_2)] = \mathbb{E}[h(q_{Y_1}(U), q_{Y_2}(U))] = \int_0^1 (1-x)^{-(1/3+1/4)} dx = \frac{12}{5}$$

where $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $h(x, y) = xy$ for all $x, y \in \mathbb{R}$. Recalling that the Beta function is given by $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ for all $x, y > 0$ and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ we can rewrite the first expression as

$$\mathbb{E}[Y_1 Y_2] = B(3/4, 2/3) = \frac{\Gamma(3/4)\Gamma(2/3)}{\Gamma(17/12)} \approx 1.87.$$

Moreover, we have

$$\begin{aligned} \text{Cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1] \mathbb{E}[Y_2] = \mathbb{E}[Y_1 Y_2] - 2 \\ &= \begin{cases} B(3/4, 2/3) - 2, & \text{if } Y_1, Y_2 \text{ are coupled by } W, \\ \frac{2}{5}, & \text{if } Y_1, Y_2 \text{ are coupled by } M, \end{cases} \end{aligned}$$

(Beta function not necessary to get 1/2 point for $\text{Cov}(Y_1, Y_2)$ in first case.) Using $\rho(Y_1, Y_2) = \text{Cov}(Y_1, Y_2) / \sqrt{\text{Var}(Y_1) \text{Var}(Y_2)}$ we finally obtain

$$\rho_{\min} = \rho(Y_1, Y_2) = \frac{B(3/4, 2/3) - 2}{\sqrt{6/36}} \approx -0.31$$

and

$$\rho_{\max} = \rho(Y_1, Y_2) = \frac{2/5}{\sqrt{6/36}} = \frac{2\sqrt{6}}{5}.$$

Question 4 (10 Pts)

a) Let (X, Y) be a two-dimensional random vector with joint distribution function

$$F(x, y) = \frac{1}{\frac{x^\alpha}{x^\alpha - 1} + e^{-y}} \quad x > 1, y \in \mathbb{R}, \alpha > 0.$$

Compute the marginal distributions and the copula of (X, Y) .

(5 Pts)

b) Let $F: \mathbb{R} \rightarrow [0, 1]$ be a cdf satisfying

$$\lim_{x \rightarrow \infty} (1 - F(x))e^{\lambda x} = b$$

for constants $\lambda, b > 0$. Does F belong to the maximum domain of attraction of a standard extreme value distribution H_ξ ? If yes, determine the shape parameter ξ and a pair of normalizing sequences. (5 Pts)

Solution 4

a) Taking the limits $x \rightarrow \infty, y \rightarrow \infty$ and noting $x^\alpha/(x^\alpha - 1) = 1/(1 - x^{-\alpha})$ we immediately see that the margins are given by

$$F_1(x) = \lim_{y \rightarrow \infty} F(x, y) = 1 - x^{-\alpha} \quad \text{and} \quad F_2(y) = \lim_{x \rightarrow \infty} F(x, y) = \frac{1}{1 + e^{-y}},$$

respectively. Since the margins F_1, F_2 are continuous Sklar's theorem ensures that the copula C of (X, Y) is unique and given by

$$C(u, v) = F(q_X(u), q_Y(v))$$

for all $u, v \in (0, 1)$, where q_X, q_Y are arbitrary quantile functions of X, Y , respectively. By inverting F_1, F_2 we obtain

$$q_X(u) = \left(\frac{1}{1-u} \right)^{1/\alpha}, \quad q_Y(v) = \log \left(\frac{v}{1-v} \right)$$

for all $u, v \in (0, 1)$. Hence,

$$C(u, v) = \frac{1}{\frac{1/(1-u)}{1/(1-u)-1} + e^{-\log(v/(1-v))}} = \frac{1}{\frac{1}{u} + \frac{1}{v} - 1}$$

for all $u, v \in (0, 1)$.

b) First note that the condition $\lim_{x \rightarrow \infty} e^{\lambda x}(1 - F(x)) = b$ is equivalent to

$$\lim_{x \rightarrow \infty} \frac{1 - F(x)}{be^{-\lambda x}} = 1$$

and therefore $1 - F(x) \sim be^{-\lambda x}$ as $x \rightarrow \infty$. Setting

$$c_n = 1/\lambda \quad d_n = \log(bn)/\lambda$$

we find

$$\begin{aligned} \lim_{n \rightarrow \infty} F^n(c_n x + d_n) &= \lim_{n \rightarrow \infty} \left(1 - \frac{1 - F(c_n x + d_n)}{be^{-\lambda(c_n x + d_n)}} be^{-\lambda(c_n x + d_n)} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1 - F\left(\frac{x + \log(bn)}{\lambda}\right)}{b \exp\left(-\lambda\left(\frac{x + \log(bn)}{\lambda}\right)\right)} \frac{e^{-x}}{n} \right)^n \\ &= \exp(-e^{-x}) \end{aligned}$$

for all $x \in \mathbb{R}$. From the second to the third line we have used that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$$

for all sequences $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$ for some $a \in \mathbb{R}$ and $1 - F(x) \sim be^{-\lambda x}$ as $x \rightarrow \infty$. Thus, F is in the maximum domain of attraction of the Gumbel distribution H_0 .

Question 5 (8 Pts)

- a) Name different stylized facts of typical daily equity log-return series. (4 Pts)
- b) Discuss and compare different methods of generating loss distributions of financial assets. (4 Pts)

Solution 5

- a) The stylized facts of univariate daily equity log-return series are:

- (U1) The series show little serial correlation;
- (U2) Series of absolute or squared log-returns show profound serial correlation;
- (U3) Conditional expected log-returns are close to zero;
- (U4) Volatility (conditional standard deviation) varies over time;
- (U5) Extreme returns appear in clusters;
- (U6) The distribution is leptokurtic or heavy-tailed (power-like tail).

The stylized facts of multivariate daily equity log-return series are:

- (M1) Multivariate log-return series show little evidence of cross-correlation, except for contemporaneous log-returns (i.e. at the same time t);
 - (M2) Multivariate series of absolute log-returns show profound cross-correlation;
 - (M3) Correlations between contemporaneous log-returns vary over time;
 - (M4) Extreme log-returns in one series often coincide with extreme log-returns in several other series.
- b) In the lecture we have seen three methods for generating loss distributions of a portfolio of assets, namely:
 - *Analytical method*: Model changes in risk factors \mathbf{X}_{t+1} and risk mappings f in such a way that the (conditional) loss distribution L_{t+1} (or its linearized form L_{t+1}^Δ) can be derived in closed form. As an example we have discussed during the lecture the variance-covariance method for market risk in which one chooses f differentiable and a iid sequence $\mathbf{X}_{t+1} \sim N_d(\mu, \Sigma)$.
 - *Historical simulation*: Approximate the loss distribution L_{t+1} by the edf $\hat{F}_L(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{\ell_{t-i+1} \leq x\}}$ where $\ell_{t-n+1}, \dots, \ell_t$ are the last n realized losses.
 - *Monte Carlo simulation*: Model L_{t+1} and simulate from it. Use simulations ℓ_1, \dots, ℓ_n to generate the simulated distribution function $\hat{F}_L(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{\ell_i \leq x\}}$.

The main advantage of the analytical method is its simplicity to implement it and the associated speed of computation. Its main disadvantage is that it is fairly limiting because the number of multivariate distributions \mathbf{X}_{t+1} and risk factor mappings f for which the loss distribution L_{t+1} can be determined analytically is quite small

The main advantage of historical simulation is that makes no modelling assumptions on the data and it does not require any estimation. The main disadvantage is that it requires a lot of data for all risk factors and predictions are based on past data.

The main advantage of a Monte Carlo simulation is its flexibility and generality as well as it does not need a lot of assumptions. In theory, any multivariate distribution can be simulated. The main drawback is that for certain distributions, simulation in high dimensions can be quite time-consuming and one needs a good model for the loss distribution L_{t+1} .