Prof. P. Cheridito September 2, 2021

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each problem on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten sides of summary are allowed

Please fill in the following table.

Last name	
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Student number (if available)	

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Question	**Points**	**Control**	**Maximum**
#1			10
#2			10
#3			10
#4			10
#5			10
Total			50

Question 1 (10 Pts)
Let L be a random loss of the form $L = YZ$, where Y is a Bernoulli random variable with mean $p \in (0,1)$ and Z an independent random variable with cdf

$$F_Z(x) = \begin{cases} 1 - x^{-\beta} & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 1 \end{cases}$$

for a parameter $\beta > 2$.

a) Compute the mean and the variance of L. (2 Pts)

b) Derive the cdf of L. (1 Pt)

c) Does L have a density? If yes, can you derive it? (1 Pt)

d) Compute $\mathrm{VaR}_\alpha(L)$ for $\alpha \in (0,1)$. (2 Pts)

e) Compute $\mathrm{ES}_\alpha(L)$ for $\alpha \in (0,1)$. (2 Pts)

f) For which $\alpha \in (0,1)$ is $\mathrm{AVaR}_\alpha(L)$ equal to $\mathrm{ES}_\alpha(L)$? (2 Pts)

Solution 1

a) Since Y and Z are independent, we have $\mathbb{E}[YZ] = \mathbb{E}[Y]\,\mathbb{E}[Z]$ and $\mathrm{Var}(YZ) = \mathbb{E}\left[Y^2\right]\mathbb{E}\left[Z^2\right] - \mathbb{E}[Y]^2\,\mathbb{E}[Z]^2$. Notice that the density function of Z is

$$f_Z(x) = \begin{cases} \beta x^{-\beta-1} & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 1. \end{cases}$$

Therefore, we can calculate the expectation as follows:

$$\mathbb{E}[Z] = \int_1^\infty x \cdot \beta x^{-\beta-1} dx = \int_1^\infty \beta x^{-\beta} dx = \frac{\beta}{-\beta+1} x^{-\beta+1}\bigg|_1^\infty = \frac{\beta}{\beta-1}$$

since $\lim_{x\to\infty} x^{-\beta+1} = 0$ when $\beta > 2$. By using the definition of expectation, we immediately get

$$\mathbb{E}[Y^2] = 1^2 \cdot \mathbb{P}[Y = 1] = p.$$

Similarly, we have

$$\mathbb{E}\left[Z^2\right] = \int_1^\infty x^2 \cdot \beta x^{-\beta-1} dx = \int_1^\infty \beta x^{-\beta+1} dx = \frac{\beta}{-\beta+2} x^{-\beta+2}\bigg|_1^\infty = \frac{\beta}{\beta-2}$$

since $\lim_{x\to\infty} x^{-\beta+2} = 0$ when $\beta > 2$. Hence, we obtain that

$$\mathbb{E}[L] = \frac{p\beta}{\beta-1} \quad \text{and} \quad \mathrm{Var}(L) = \frac{p\beta}{\beta-2} - \frac{p^2\beta^2}{(\beta-1)^2}.$$

b) We have

$$L = YZ = \begin{cases} Z & \text{if} \quad Y = 1, \\ 0 & \text{if} \quad Y = 0. \end{cases}$$

So, the cdf of L is given by

$$\begin{aligned}
F_L(x) &= \mathbb{P}[L \le x] = \mathbb{P}[YZ \le x] = \mathbb{P}[YZ \le x \mid Y = 1]\mathbb{P}[Y = 1] + \mathbb{P}[YZ \le x \mid Y = 0]\mathbb{P}[Y = 0] \\
&= \mathbb{P}[Z \le x]\mathbb{P}[Y = 1] + \mathbb{P}[0 \le x]\mathbb{P}[Y = 0] = p \cdot \mathbb{P}[Z \le x] + (1-p) \cdot \mathbb{P}[0 \le x] \\
&= \begin{cases} p \cdot \mathbb{P}[Z \le x] + (1-p) & \text{if} \quad x \ge 0, \\ 0 & \text{if} \quad x < 0, \end{cases} \\
&= \begin{cases} p(1 - x^{-\beta}) + (1-p) & \text{if} \quad x \ge 1, \\ 1-p & \text{if} \quad 0 \le x < 1, \\ 0 & \text{if} \quad x < 0, \end{cases} \\
&= \begin{cases} 1 - px^{-\beta} & \text{if} \quad x \ge 1, \\ 1-p & \text{if} \quad 0 \le x < 1, \\ 0 & \text{if} \quad x < 0. \end{cases}
\end{aligned}$$

c) L cannot have a density because its cdf F_L jumps at 0. (Note that the jump size $1-p$ equals the probability that L is equal to 0.)

d) If $1 - p < \alpha < 1$, $\mathrm{VaR}_\alpha(L)$ is equal to the unique $x > 1$ satisfying $1 - px^{-\beta} = \alpha$, whereas for $0 < \alpha \le 1 - p$, one has $\mathrm{VaR}_\alpha(L) = 0$. Therefore,

$$\mathrm{VaR}_\alpha(L) = \begin{cases} \left(\frac{p}{1-\alpha}\right)^{1/\beta} & \text{if} \quad \alpha \in (1-p, 1) \\ 0 & \text{if} \quad \alpha \in (0, 1-p]. \end{cases}$$

e) If $1 - p < \alpha < 1$, $\mathrm{VaR}_\alpha(L) = \left(\frac{p}{1-\alpha}\right)^{1/\beta} > 1$. So, since

$$1_{\{x \ge 1\}} \beta x^{-\beta-1}$$

is the density of Z, we obtain

$$\begin{aligned}
\mathrm{ES}_\alpha(L) &= \mathbb{E}\left[L \mid L \ge \mathrm{VaR}_\alpha(L)\right] = \frac{1}{\mathbb{P}[L \ge \mathrm{VaR}_\alpha(L)]} \int_{\mathrm{VaR}_\alpha(L)}^\infty x \, dF_L(x) \\
&= \frac{1}{1-\alpha} \int_{\mathrm{VaR}_\alpha(L)}^\infty x \cdot p\beta x^{-\beta-1} dx = \frac{1}{1-\alpha} \int_{\mathrm{VaR}_\alpha(L)}^\infty p\beta x^{-\beta} dx \\
&= \frac{1}{1-\alpha} \cdot \frac{p\beta}{-\beta+1} x^{-\beta+1} \Big|_{\mathrm{VaR}_\alpha(L)}^\infty = \frac{1}{1-\alpha} \cdot \frac{p\beta}{\beta-1} \mathrm{VaR}_\alpha(L)^{-\beta+1} \\
&= \frac{\beta}{\beta-1} \left(\frac{p}{1-\alpha}\right)^{1/\beta}
\end{aligned}$$

because $\lim_{x\to\infty} x^{-\beta+1} = 0$ when $\beta > 2$.

On the other hand, if $0 < \alpha \le 1 - p$, then $\mathrm{VaR}_\alpha(L) = 0$ and

$$\mathbb{P}[L \ge \mathrm{VaR}_\alpha(L)] = \mathbb{P}[L \ge 0] = 1.$$

Therefore, we have

$$\mathrm{ES}_\alpha(L) = \frac{1}{\mathbb{P}[L \ge \mathrm{VaR}_\alpha(L)]} \int_{\mathrm{VaR}_\alpha(L)}^\infty x \, dF_L(x) = \int_0^\infty x \, dF_L(x) = \mathbb{E}\left[L\right] = \frac{p\beta}{\beta-1}.$$

So,

$$\mathrm{ES}_\alpha(L) = \begin{cases} \frac{\beta}{\beta-1}\left(\frac{p}{1-\alpha}\right)^{1/\beta} & \text{if} \quad \alpha \in (1-p,1) \\ \frac{p\beta}{\beta-1} & \text{if} \quad \alpha \in (0,1-p]. \end{cases}$$

f) We know from the lecture notes that $\mathrm{AVaR}_\alpha(L) = \mathrm{ES}_\alpha(L)$ if and only if $\mathbb{P}[L \geq \mathrm{VaR}_\alpha(L)] = 1-\alpha$.

Since

$$\mathbb{P}[L \geq \mathrm{VaR}_\alpha(L)] = \begin{cases} 1-\alpha & \text{if} \quad \alpha \in (1-p,1) \\ 1 & \text{if} \quad \alpha \in (0,1-p], \end{cases}$$

we have $\mathrm{AVaR}_\alpha(L) = \mathrm{ES}_\alpha(L)$ if and only if $\alpha \in (1-p,1)$.

Alternatively, with the definition of $\mathrm{AVaR}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)du$, we can compute $\mathrm{AVaR}_\alpha(L)$ and then compare its value with $\mathrm{ES}_\alpha(L)$ in e). If $1-p<\alpha<1$, we obtain from the change of variable $y=1-u$,

$$\begin{aligned} \mathrm{AVaR}_\alpha(L) &= \frac{1}{1-\alpha}\int_\alpha^1 \left(\frac{p}{1-u}\right)^{1/\beta} du \\ &= \frac{1}{1-\alpha}\int_0^{1-\alpha} \left(\frac{p}{y}\right)^{1/\beta} dy = \frac{p^{1/\beta}}{1-\alpha} \cdot \frac{y^{-1/\beta+1}}{-1/\beta+1}\Bigg|_0^{1-\alpha} \\ &= \frac{\beta}{\beta-1}\left(\frac{p}{1-\alpha}\right)^{1/\beta} = \mathrm{ES}_\alpha(L). \end{aligned}$$

If $0<\alpha\leq 1-p$, with the change of variable $y=1-u$ again, we have

$$\begin{aligned} \mathrm{AVaR}_\alpha(L) &= \frac{1}{1-\alpha}\left(\int_\alpha^{1-p} \mathrm{VaR}_u(L)du + \int_{1-p}^1 \mathrm{VaR}_u(L)du\right) \\ &= \frac{1}{1-\alpha}\left(\int_\alpha^{1-p} 0\,du + \int_{1-p}^1 \left(\frac{p}{1-u}\right)^{1/\beta} du\right) \\ &= \frac{1}{1-\alpha}\int_0^{p} \left(\frac{p}{y}\right)^{1/\beta} dy = \frac{p^{1/\beta}}{1-\alpha} \cdot \frac{y^{-1/\beta+1}}{-1/\beta+1}\Bigg|_0^{p} \\ &= \frac{1}{1-\alpha} \cdot \frac{p\beta}{\beta-1} > \mathrm{ES}_\alpha(L). \end{aligned}$$

Therefore, we conclude that $\mathrm{AVaR}_\alpha(L) = \mathrm{ES}_\alpha(L)$ if and only if $\alpha \in (1-p,1)$.

Question 2 (10 Pts)

a) Consider a d-dimensional random vector $X = (X_1, \dots, X_d) \sim N_d(\mu, \Sigma)$ such that $X_1 \equiv 1$. Denote by $\mathcal{L}$ the set of random losses $\{v^T X : v \in \mathbb{R}^d\}$ and let $\alpha \in [1/2, 1)$. Which properties of a coherent risk measure does the mapping $\mathrm{VaR}_\alpha \colon \mathcal{L} \to \mathbb{R}$ have? Explain your answers. (5 Pts)

b) Assume d financial returns are described by the components of a d-dimensional random vector $X = (X_1, \dots, X_d)$ with an elliptical distribution such that $\mathbb{E}[X_i^2] < \infty$ for all $i = 1, \dots, d$. Let $v, w \in \mathbb{R}^d$ be two portfolio vectors such that $v^T \mu = w^T \mu$, where $\mu \in \mathbb{R}^d$ is the mean vector of X. Show that, for all $\alpha \in [1/2, 1)$, one has

$$\mathrm{ES}_\alpha\left(-v^T X\right) \le \mathrm{ES}_\alpha\left(-w^T X\right) \quad \text{if and only if} \quad \mathrm{Var}\left(v^T X\right) \le \mathrm{Var}\left(w^T X\right).$$

(5 Pts)

Solution 2

a) Known from the lecture, VaR_α satisfies monotonicity (M), translation property (T) and positive homogeneity (P), hence we only need to check subadditivity (S), that is $\mathrm{VaR}_\alpha(L_1 + L_2) \le \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)$. For $j \in \{1, 2\}$, let $L_j = v_j^T X \in \mathcal{L}$ denote two losses from $\mathcal{L}$. As a linear combination of normal distribution, we know $L_j \sim N_d(\mu_{L_j}, \sigma_{L_j}^2)$ where $\mu_{L_j} = v_j^T \mu$ and $\sigma_{L_j}^2 = v_j^T \Sigma v_j = \mathrm{Var}(v_j^T X) = \mathrm{Var}(L_j)$. Furthermore, $L_1 + L_2 = (v_1 + v_2)^T X \in \mathcal{L}$ and $L_1 + L_2 \sim N_d(\mu_{L_1+L_2}, \sigma_{L_1+L_2}^2)$ where $\mu_{L_1+L_2} = \mu_{L_1} + \mu_{L_2}$. Using the Cauchy-Schwarz inequality, we have $\mathrm{Cov}\,(L_1, L_2) \le \sqrt{\mathrm{Var}(L_1)\mathrm{Var}(L_2)}$. Therefore, we have

$$\begin{aligned} \sigma_{L_1+L_2}^2 &= \mathrm{Var}(L_1 + L_2) = \mathrm{Var}(L_1) + \mathrm{Var}(L_2) + 2\,\mathrm{Cov}\,(L_1, L_2) \\ &\le \mathrm{Var}(L_1) + \mathrm{Var}(L_2) + 2\sqrt{\mathrm{Var}(L_1)\mathrm{Var}(L_2)} = (\sigma_{L_1} + \sigma_{L_2})^2. \end{aligned}$$

Hence, $\sigma_{L_1+L_2} \le \sigma_{L_1} + \sigma_{L_2}$. Notice that for $\alpha \in [1/2, 1)$, $\Phi^{-1}(\alpha) \ge 0$ where Φ is the cdf of d-dimensional standard normal distribution. One then has

$$\begin{aligned} \mathrm{VaR}_\alpha(L_1 + L_2) &= \mu_{L_1+L_2} + \sigma_{L_1+L_2}\Phi^{-1}(\alpha) = \mu_{L_1} + \mu_{L_2} + \sigma_{L_1+L_2}\Phi^{-1}(\alpha) \\ &\le \mu_{L_1} + \mu_{L_2} + (\sigma_{L_1} + \sigma_{L_2})\Phi^{-1}(\alpha) \\ &= \mu_{L_1} + \sigma_{L_1}\Phi^{-1}(\alpha) + \mu_{L_2} + \sigma_{L_2}\Phi^{-1}(\alpha) = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2). \end{aligned}$$

Thus, VaR_α also satisfies subadditivity (S) hence satisfies all properties of a coherent risk measure.

b) Since X has an elliptical distribution with mean vector $\mu \in \mathbb{R}^d$, we then have $X \overset{(d)}{=} \mu + AY \sim E_d(\mu, \Sigma, \psi)$ where $A \in \mathbb{R}^{d \times k}$, $AA^T = \Sigma$ and $Y \sim S_k(\psi)$.Notice that $-v^T X \overset{(d)}{=} -v^T(\mu + AY) = -v^T\mu - v^T AY$, according to the lecture notes, we know that

$$-v^T AY = (-A^T v)^T Y \overset{(d)}{=} \| - A^T v\| Y_1 = \|A^T v\| Y_1.$$

Since ES_α is distribution-based and satisfies translation property (T) and positive homogeneity (P), we have $\mathrm{ES}_\alpha\left(-v^T X\right) = -v^T\mu + \|A^T v\| \mathrm{ES}_\alpha\left(Y_1\right)$. Similarly, we have

$\mathrm{ES}_\alpha\left(-w^T X\right) = -w^T\mu + \|A^T w\|\mathrm{ES}_\alpha\left(Y_1\right)$. We further notice that for all $\alpha \in [1/2, 1)$, $\mathrm{ES}_\alpha\left(Y_1\right) \geq 0$. Using the fact that $v^T\mu = w^T\mu$, one deduces

$$\begin{aligned}
\mathrm{ES}_\alpha\left(-v^T X\right) \leq \mathrm{ES}_\alpha\left(-w^T X\right) \quad &\Leftrightarrow \|A^T v\|\mathrm{ES}_\alpha\left(Y_1\right) \leq \|A^T w\|\mathrm{ES}_\alpha\left(Y_1\right) \\
&\Leftrightarrow \|A^T v\| \leq \|A^T w\| \\
&\Leftrightarrow v^T A A^T v \leq w^T A A^T w \\
&\Leftrightarrow v^T \Sigma v \leq w^T \Sigma w.
\end{aligned}$$

We further notice that

$$\mathrm{Var}\left(v^T X\right) = \mathrm{Var}\left(v^T A Y\right) = \mathrm{Var}\left(\|A^T v\| Y_1\right) = \|A^T v\|^2 \mathrm{Var}\left(Y_1\right) = v^T \Sigma v \mathrm{Var}\left(Y_1\right).$$

Similarly, we have $\mathrm{Var}\left(w^T X\right) = w^T \Sigma w \mathrm{Var}\left(Y_1\right)$. Since $\mathrm{Var}\left(Y_1\right) \geq 0$, one could further deduce

$$v^T \Sigma v \leq w^T \Sigma w \Leftrightarrow \mathrm{Var}\left(v^T X\right) \leq \mathrm{Var}\left(w^T X\right)$$

and hence finally conclude that

$$\mathrm{ES}_\alpha\left(-v^T X\right) \leq \mathrm{ES}_\alpha\left(-w^T X\right) \Leftrightarrow \mathrm{Var}\left(v^T X\right) \leq \mathrm{Var}\left(w^T X\right).$$

Question 3 (10 Pts)
Let X be a non-negative random variable with cdf

$$F(x) = 1 - \frac{1}{\sqrt{1+2x}}, \quad x \geq 0.$$

a) Does X have a density? If yes, can you derive it? (1 Pt)

b) Find all $k \in \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$? (1 Pt)

c) Does F belong to the maximum domain of attraction of a standard generalized extreme value distribution H_ξ? If yes, determine the shape parameter ξ and a pair of normalizing sequences. (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$, over a threshold $u > 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u \to \infty} \sup_{x > 0} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0,$$

for a generalized Pareto distribution $G_{\xi,\beta(u)}$? If yes, for which ξ and $\beta(u)$ does this hold? (3 Pts)

Solution 3

a) Yes. Since the cdf F is smooth on $[0, \infty)$ its pdf is given by

$$f(x) = \frac{dF}{dx}(x) = \frac{1}{(1+2x)^{3/2}}$$

for all $x \geq 0$ and otherwise vanishes.

b) Since the support of the distribution is $[0, \infty)$, we have $\mathbb{E}[|X|^k] = \mathbb{E}[X^k]$. With the density function we obtained in a), we calculate

$$\mathbb{E}[|X|^k] = \mathbb{E}[X^k] = \int_0^\infty \frac{x^k}{(1+2x)^{3/2}} dx.$$

This integral diverges for all $k \geq 1$, so there is no $k \in \mathbb{N} = \{1, 2, \dots\}$ for which $\mathbb{E}[|X|^k] < \infty$.

c) One has

$$\overline{F}(x) = 1 - F(x) = \tfrac{1}{\sqrt{1+2x}} = x^{-1/2} \sqrt{\tfrac{x}{1+2x}} = x^{-1/2} L(x)$$

for all $x \geq 0$, where $L(x) = \sqrt{1/(2 + x^{-1})}$. The function $L(x)$ satisfies

$$\lim_{x \to \infty} \tfrac{L(tx)}{L(x)} = \frac{\sqrt{2+\frac{1}{x}}}{\sqrt{2+\frac{1}{tx}}} = 1$$

for all $t > 0$ and therefore it is a slowly varying function.

Hence, by the characterization of the distributions in the maximum domain of attraction of a Fréchet distribution provided in the lecture, we deduce $F \in \text{MDA}(H_2)$, where H_2 is given by

$$H_2(x) = \begin{cases} \exp\left(-\frac{1}{\sqrt{1+2x}}\right) & \text{if } x > -1/2, \\ 0 & \text{if } x \le -1/2. \end{cases}$$

Note that

$$\lim_{n\to\infty} (1 + \tfrac{x}{n})^n = \exp(x) \text{ for all } x \in \mathbb{R}.$$

So for $c_n = n^2$ and $d_n = (n^2 - 1)/2$, one obtains

$$\begin{aligned} \lim_{n\to\infty} F^n(c_n x + d_n) &= \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{1+2(n^2x+(n^2-1)/2)}}\right)^n \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n\sqrt{1+2x}}\right)^n = \exp\left(-\frac{1}{\sqrt{1+2x}}\right) = H_2(x) \end{aligned}$$

for $x > -1/2$, and

$$\lim_{n\to\infty} F^n(c_n x + d_n) = 0 = H_2(x)$$

for $x \le -1/2$.

d) We have for $x \ge 0$ and $u > 0$,

$$F_u(x) = \frac{F(x+u) - F(u)}{1 - F(u)}.$$

It gives

$$F_u(x) = \frac{[1+2u]^{-1/2} - [1+2(x+u)]^{-1/2}}{[1+2u]^{-1/2}} = 1 - \frac{\sqrt{1+2u}}{\sqrt{1+2(x+u)}}.$$

e) Pickands-Balkema-de Haan theorem gives us that

$$\lim_{u\to\infty} \sup_{x>0} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0, \tag{1}$$

if and only if $F \in MDA(H_\xi)$. We have shown in (c) that $F \in MDA(H_2)$, thus (1) holds for $\xi = 2$ and for some function $\beta(u)$ yet to be determined.

We have that

$$\begin{aligned} |F_u(x) - G_{2,\beta(u)}(x)| &= \left| 1 - \frac{\sqrt{1+2u}}{\sqrt{1+2(x+u)}} - 1 + \left(1 + \frac{2x}{\beta(u)}\right)^{-1/2} \right| \\ &= \left| \frac{\sqrt{\beta(u)}}{\sqrt{2x+\beta(u)}} - \frac{\sqrt{1+2u}}{\sqrt{2x+1+2u}} \right|, \end{aligned}$$

which is equal to 0 for $\beta(u) = 2u + 1$. This choice of beta will also render the limit in (1) equal to 0.

Question 4 (10 Pts)
Let (X,Y) be a two-dimensional random vector with cdf

$$F_{X,Y}(x,y) = \frac{(\sqrt{1+2x}-1)\,(1-e^{-4y^2})}{\sqrt{1+2x}-\frac{1}{2}e^{-4y^2}}, \quad x,y \geq 0.$$

a) What are the marginal distributions of X and Y? (3 Pts)

b) Compute a copula C of (X,Y). Is it unique? (3 Pts)

c) Calculate the coefficient of upper tail dependence λ_u between X and Y. (2 Pts)

d) Calculate the coefficient of lower tail dependence λ_l between X and Y. (2 Pts)

Solution 4

a) Taking the limits $x \to \infty$ and $y \to \infty$ we see that the margins are given by

$$\begin{aligned} F_X(x) &= \lim_{y\to\infty} F_{X,Y}(x,y) = \lim_{y\to\infty} \frac{(\sqrt{1+2x}-1)\,(1-e^{-4y^2})}{\sqrt{1+2x}-\frac{1}{2}e^{-4y^2}} \\ &= \frac{\sqrt{1+2x}-1}{\sqrt{1+2x}} = 1-\frac{1}{\sqrt{1+2x}}, \text{ for } x \geq 0 \end{aligned}$$

and by using l'Hôpital's rule

$$\begin{aligned} F_Y(y) &= \lim_{x\to\infty} F_{X,Y}(x,y) = \lim_{x\to\infty} \frac{(\sqrt{1+2x}-1)\,(1-e^{-4y^2})}{\sqrt{1+2x}-\frac{1}{2}e^{-4y^2}} \\ &= \lim_{x\to\infty} \frac{\sqrt{1+2x}\,(1-e^{-4y^2})}{\sqrt{1+2x}} = 1-e^{-4y^2}, \text{ for } y \geq 0. \end{aligned}$$

b) Since the marginals are continuous, Sklar's theorem ensures that the copula of (X,Y) is unique and given by

$$C(u,v) = F_{X,Y}(q_X(u), q_Y(v)) \text{ for } u,v \in (0,1)^2.$$

By inverting F_X and F_Y we obtain the quantile functions as

$$q_X(u) = \frac{u(2-u)}{2(1-u)^2}.$$

and

$$q_Y(v) = \frac{1}{2}\left(\log\left(\frac{1}{1-v}\right)\right)^{1/2}$$

Therefore, one deduces

$$\begin{aligned} C(u,v) &= F_{X,Y}\left(\frac{u(2-u)}{2(1-u)^2}, \frac{1}{2}\left(\log\left(\frac{1}{1-v}\right)\right)^{1/2}\right) \\ &= \frac{uv}{1-1/2(1-u)(1-v)}, \qquad \text{for } u,v \in (0,1)^2. \end{aligned}$$

c) We have seen in the lecture that the coefficient of upper tail dependence can be computed as

$$\lambda_u = \lim_{\alpha \uparrow 1} \frac{1 - 2\alpha + C(\alpha, \alpha)}{1 - \alpha}.$$

Using the result from b), one has

$$C(\alpha, \alpha) = \frac{\alpha^2}{1 - \frac{1}{2}(1-\alpha)^2}.$$

Therefore,

$$\begin{aligned}
\lambda_u &= \lim_{\alpha \uparrow 1} \frac{1 - 2\alpha + \frac{\alpha^2}{1-\frac{1}{2}(1-\alpha)^2}}{1-\alpha} = \lim_{\alpha \uparrow 1} \frac{(1-2\alpha)\left(1 - \frac{1}{2}(1-\alpha)^2\right) + \alpha^2}{(1-\alpha)\left(1 - \frac{1}{2}(1-\alpha)^2\right)} \\
&= \lim_{\alpha \uparrow 1} \frac{1 - 2\alpha + \alpha^2 - \frac{1}{2}(1-\alpha)^2(1-2\alpha)}{(1-\alpha)\left(1 - \frac{1}{2}(1-\alpha)^2\right)} = \lim_{\alpha \uparrow 1} \frac{(1-\alpha)^2\left(1 - \frac{1}{2}(1-2\alpha)\right)}{(1-\alpha)\left(1 - \frac{1}{2}(1-\alpha)^2\right)} \\
&= \lim_{\alpha \uparrow 1} \frac{(1-\alpha)\left(1 - \frac{1}{2}(1-2\alpha)\right)}{\left(1 - \frac{1}{2}(1-\alpha)^2\right)} = \lim_{\alpha \uparrow 1} \frac{(1-\alpha)\left(\frac{1}{2} + \alpha\right)}{1/2 + \alpha - \alpha^2/2} = 0.
\end{aligned}$$

Alternatively, the coefficient of upper tail dependence can be computed as

$$\lambda_u = 2 - \lim_{\alpha \uparrow 1} \frac{1 - C(\alpha, \alpha)}{1 - \alpha}.$$

where

$$C(\alpha, \alpha) = \frac{\alpha^2}{1 - \frac{1}{2}(1-\alpha)^2}.$$

By using l'Hôpital's rule,

$$\begin{aligned}
&\lim_{\alpha \uparrow 1} \frac{1 - C(\alpha, \alpha)}{1 - \alpha} = \lim_{\alpha \uparrow 1} \frac{d}{d\alpha} C(\alpha, \alpha) \\
&= \lim_{\alpha \uparrow 1} \frac{2\alpha\left(1 - 1/2(1-\alpha)^2\right) - \alpha^2(1-\alpha)}{\left(1 - 1/2(1-\alpha)^2\right)^2} = \frac{2 - 0}{1} = 2.
\end{aligned}$$

Hence, $\lambda_u = 0$.

d) We know from the lecture that the coefficient of lower tail dependence can be computed as

$$\lambda_l = \lim_{\alpha \downarrow 0} \frac{C(\alpha, \alpha)}{\alpha}.$$

So using the result from b),

$$C(\alpha, \alpha) = \frac{\alpha^2}{1 - \frac{1}{2}(1-\alpha)^2},$$

one has

$$\lambda_l = \lim_{\alpha \downarrow 0} \frac{\alpha}{1 - \frac{1}{2}(1-\alpha)^2} = 0.$$

Question 5 (10 Pts)

a) Why is subadditivity a desirable property of a risk measure? (2 Pts)

b) Why does one usually assume stationarity in time series modelling? (2 Pts)

c) How can a multivariate t-distribution be represented as a normal mixture distribution? (3 Pts)

d) Name advantages and disadvantages of elliptical distributions in financial modelling. (3 Pts)

Solution 5

a) There are several properties of subadditivity that make it a desirable property of a risk measure. For instance:

- Subadditivity is consistent with the concept that diversification reduces risk.
- A subadditive risk measure can detect a concentration of risk.
- Subadditivity permits decentralized risk measurement by sub-units of a firm and removes any incentive to split the firm to reduce capital requirements. For example to bound $\rho(L_1+L_2)$ by a constant c, it suffices to bound $\rho(L_j)$ by c_j, $j \in \{1,2\}$, for $c_1 + c_2 \leq c$, since, by subadditivity, $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2) \leq c_1 + c_2 \leq c$.

b) In (univariate) time series modelling, we observe realizations of a discrete-time stochastic process $(X_t)_{t \in \mathbb{Z}}$. It implies that we have only access to an outcome of a single experiment over time, albeit observations at different time points are typically not independent of each other. Think of observing realizations of $\{X_1, \ldots, X_n\}$, where each random variable is from a potentially different distribution. Since it is impossible to determine a distribution using a single observation, stationarity assumptions are needed to infer properties of the distribution of future events from historical observations. Furthermore, stationarity assumptions are required so that statistics such as expected values, variances and covariances can be estimated using time-averages. For example, a covariance function at lag one can only be estimated based on observations from $\{X_1, \ldots, X_n\}$ if (X_1, X_2), (X_2, X_3), $\ldots$, (X_{n-1}, X_n) have the same covariance.

c) Let $X \sim t_d(\nu, \mu, \Sigma)$, where $\mu \in \mathbb{R}^d$ and $\Sigma = AA^T \in \mathbb{R}^{d \times d}$. From the lecture notes we know that $X = \mu + \sqrt{W} AZ \sim M_d(\mu, \Sigma, \hat{F}_W)$, where $Z = (Z_1, \ldots, Z_d) \sim N_d(0, I_d)$ is independent of $W = 1/G$ for $G \sim \Gamma(\nu/2, \nu/2)$, or equivalently, $W \sim IG(\nu/2, \nu/2)$, or equivalently, $W = \nu/V$ for $V \sim \chi^2_\nu$.

d) Advantages:

- Affine transformations of elliptical random vectors remain elliptical with the same characteristic generator. That is, let $X \sim E_d(\mu, \Sigma, \psi)$ and take $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$. It can be shown that $b + BX \sim E_k(B\mu + b, B\Sigma B^T, \psi)$.
- Marginal distributions are elliptical with the same characteristic generator. That is, if $X = (X_1, X_2) \sim E_d(\mu, \Sigma, \psi)$, then $X_1 \sim E_k(\mu_1, \Sigma_{11}, \psi)$ and $X_2 \sim E_{d-k}(\mu_1, \Sigma_{22}, \psi)$,

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

- Conditional distributions are elliptical, albeit with a different characteristic generator.
- Quadratic forms are known.
- The sum of two independent elliptical vectors with the same dispersion matrix is also elliptical. That is, if $X \sim E_d(\mu, \Sigma, \psi)$ and $Y \sim E_d(\tilde{\mu}, c\Sigma, \tilde{\psi})$ are independent, then $aX + bY$ is elliptical.
- Subadditivity of VaR in elliptical models.
- Standard estimators of the mean vector and covariance matrix are consistent under weak assumptions, albeit they are not necessarily the best estimators of locations and dispersion for a given finite sample of elliptical data.
- Existence of various robust and efficient estimators (e.g., M-estimators) of location and dispersion.
- Mutual fund separation theorems and the Capital Asset Pricing Model (CAPM) hold for all elliptical distributions.
- It generalize the multivariate normal distribution, and many of the nice properties of the multivariate normal are preserved.
- Elliptical distributions allow both lighter-than-normal and heavier-than-normal tails which are common in financial data.

Disadvantages:

- As normal variance mixtures, elliptical distributions are radially symmetric, which is not a desirable property in the context of financial modelling.
- Traditional Gaussian methods become invalid for some problems under elliptical distributions.
- Extending the normal-theory standard procedures to the elliptical case may be much more difficult.