Statistical Sciences
UNIVERSITY OF TORONTO

Risk Bounds under Uncertainty and Model Risk

Silvana M. Pesenti

silvana.pesenti@utoronto.ca

Bahnhofskolloquium, 2. March 2020

Risk Measures

Risk assessment using risk measures

A risk measure $\rho \colon \mathcal{X} \to \mathbb{R}$ is a function mapping random variables to real numbers.

Applications in finance and insurance:

- regulatory capital requirement
- capital allocation
- insurance pricing
- ...

VaR, RVAR, ES

For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

Value-at-Risk:

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

VaR, RVAR, ES

For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

Value-at-Risk:

$$\mathsf{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

Range-Value-at-Risk:

$$\mathsf{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_\alpha^\beta \mathsf{VaR}_u(X) \mathrm{d}u.$$

VaR, RVAR, ES

For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

Value-at-Risk:

$$\mathsf{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

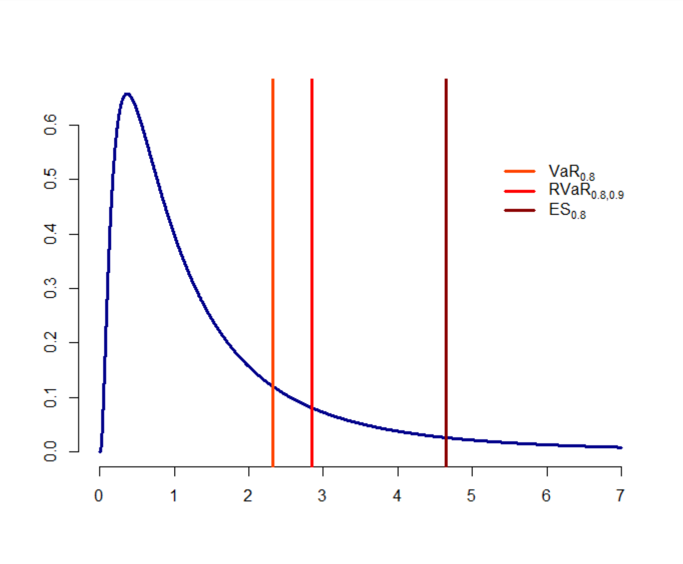
Range-Value-at-Risk:

$$\mathsf{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_\alpha^\beta \mathsf{VaR}_u(X) \mathrm{d}u.$$

Expected Shortfall:

$$\mathsf{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \mathsf{VaR}_u(X) \mathrm{d}u.$$

VaR, RVAR, ES



Risk assessment through risk measures

Properties for risk assessment:
[Artzner et al., 1999, Föllmer & Schied, 2011]
law-invariant, monotone, convex, sub-additive, coherent, translation invariant, ...

Statistical properties:
[Gneiting, 2011, Krätschmer et al., 2014, Pesenti et al., 2016]
elicitable, backtestable, robust, ...

Risk assessment under uncertainty:
[Embrechts et al., 2015, Puccetti & Rüschendorf, 2012, Wang & Wang, 2011]
bounds for risk measures, worst-case risk measures, aggregation robustness, rearrangement algorithm, joint mixability, ...

Distributional uncertainty

Distributional uncertainty

Risk assessment in the presence of uncertainty:

- distributional uncertainty
- parameter uncertainty
- distributional misspecifications
- data collection

What are the possible values of

$$\rho(X), \quad \text{if } X \in \mathcal{M},$$

for an uncertainty set $\mathcal{M}$.

Distributional uncertainty

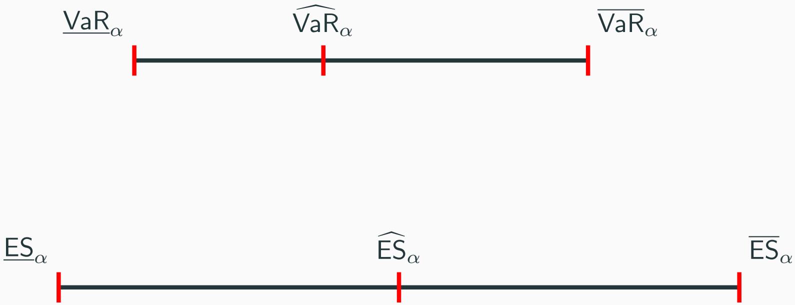
Best-case and *worst-case* risk measures

$$\underline{\rho(X)} = \inf_{X \in \mathcal{M}} \rho(X), \qquad \overline{\rho(X)} = \sup_{X \in \mathcal{M}} \rho(X).$$

Risk measure bounds:

$$\rho(X) \in \big(\underline{\rho(X)}, \overline{\rho(X)}\big)$$

VaR and ES



$\underline{\mathrm{VaR}}_\alpha$
$\widehat{\mathrm{VaR}}_\alpha$
$\overline{\mathrm{VaR}}_\alpha$
$\underline{\mathrm{ES}}_\alpha$
$\widehat{\mathrm{ES}}_\alpha$
$\overline{\mathrm{ES}}_\alpha$

Uncertainty set

An uncertainty set $\mathcal{M}$ describes the knowledge about the uncertainty in the distribution of X.

Uncertainty set

An uncertainty set $\mathcal{M}$ describes the knowledge about the uncertainty in the distribution of X.

For example: "(nearly) complete uncertainty"

$$\mathcal{M}(\mu, \sigma) = \Big\{ X \mid E(X) = \mu, \mathsf{Var}(X) = \sigma^2 \Big\}$$

Bounds with moment constraints

$\mathrm{VaR}_\alpha(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathrm{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathrm{ES}_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

Bounds with moment constraints

$\mathsf{VaR}_\alpha(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{ES}_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

! extremely large

Bounds with moment constraints

$\mathsf{VaR}_\alpha(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{ES}_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

! extremely large ! independent of the distribution of X

Bounds with moment constraints

$\mathsf{VaR}_\alpha(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\mathsf{ES}_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

! extremely large ! independent of the distribution of X

! worst-case distribution is a two point distribution.

Bounds with moment constraints

	$\underline{\rho(X)}$	$\rho(X)$		$\overline{\rho(X)}$
		Normal	Log-Normal	
$\text{VaR}_{0.975}$	9.68	13.92	14.46	22.49
$\text{RVaR}_{0.95,0.99}$	9.80	13.82	14.33	18.72
$\text{ES}_{0.95}$	10.00	14.13	14.79	18.72

X has mean 10 and standard deviation 2.

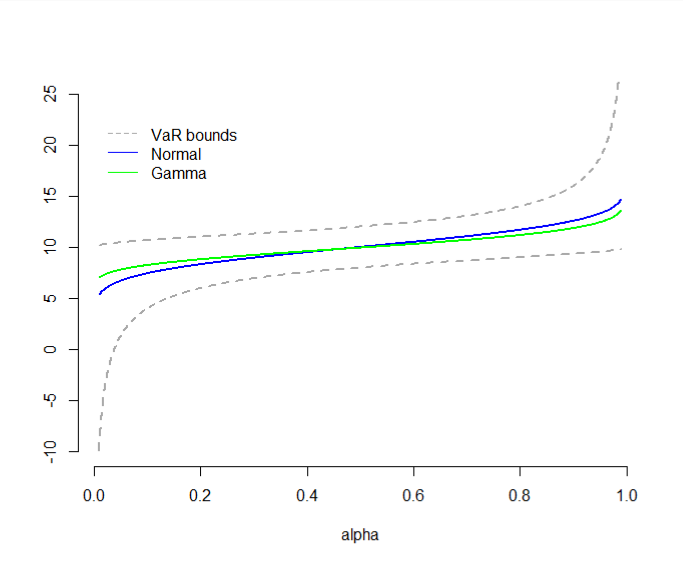
Bounds with moment constraints

	$\underline{\rho(X)}$	$\rho(X)$		$\overline{\rho(X)}$
		Normal	Log-Normal	
$\text{VaR}_{0.975}$	9.68	13.92	14.46	22.49
$\text{RVaR}_{0.95,0.99}$	9.80	13.82	14.33	18.72
$\text{ES}_{0.95}$	10.00	14.13	14.79	18.72

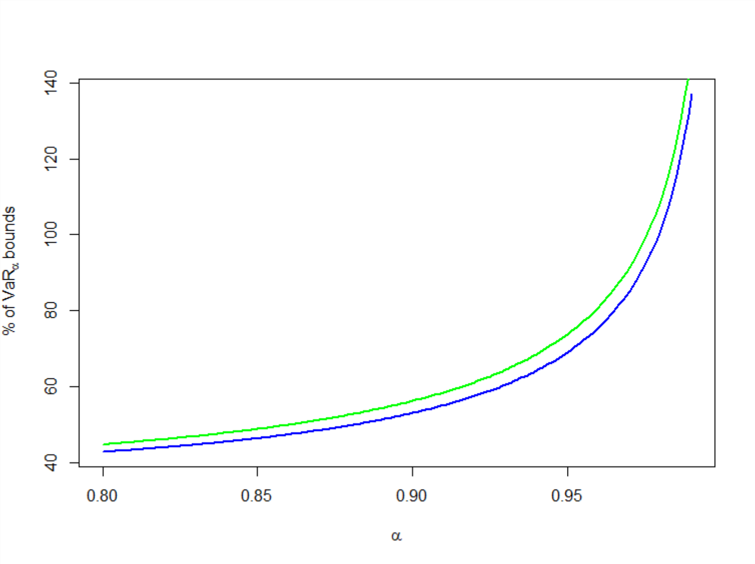
X has mean 10 and standard deviation 2.

$\Rightarrow$ For any random variable, with mean $=$ 10 and sd $=$ 2, its VaR at level 0.975 belongs to (9.68, 22.49).

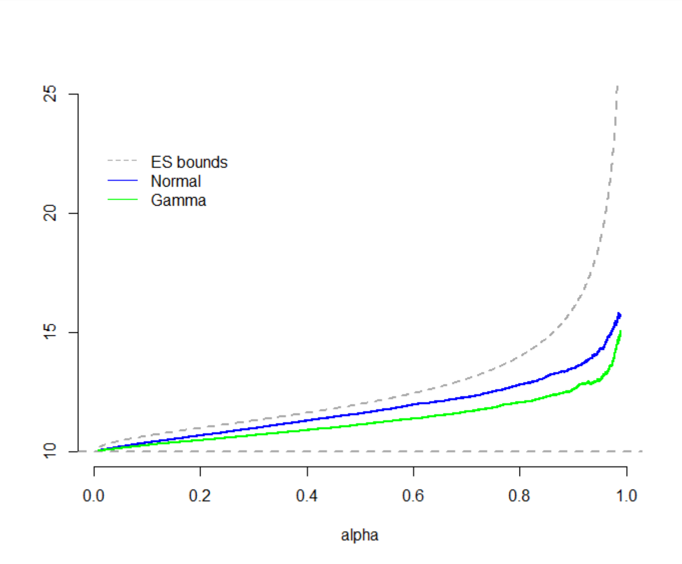
VaR bounds; with mean 10 and sd 2



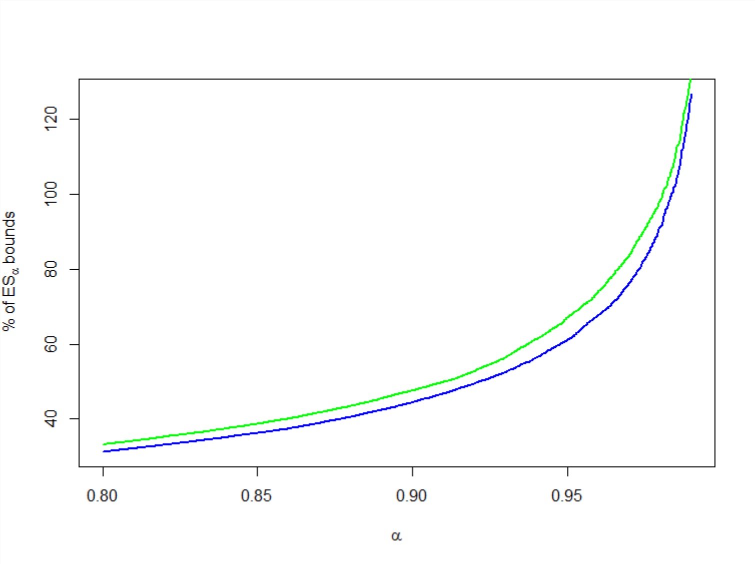
% of VaR bounds; with mean 10 and sd 2



ES bounds; with mean 10 and sd 2



% of ES bounds; with mean 10 and sd 2



Towards better bounds:

$\Rightarrow$ Include further knowledge to the uncertainty set $\mathcal{M}$.

Towards better bounds:

$\Rightarrow$ Include further knowledge to the uncertainty set $\mathcal{M}$.

- higher moments [Cornilly et al., 2018]
- symmetric distributions [Zhu & Shao, 2018, Li et al., 2018]
- unimodal distributions [Li et al., 2018].

Towards better bounds:

⇒ Include further knowledge to the uncertainty set $\mathcal{M}$.

- higher moments [Cornilly et al., 2018]
- symmetric distributions [Zhu & Shao, 2018, Li et al., 2018]
- unimodal distributions [Li et al., 2018].

⇒ only marginal improvements

⇒ worst-case distribution is a two point distribution

Towards better bounds:

⇒ Include further knowledge to the uncertainty set $\mathcal{M}$.

- higher moments [Cornilly et al., 2018]
- symmetric distributions [Zhu & Shao, 2018, Li et al., 2018]
- unimodal distributions [Li et al., 2018].

⇒ only marginal improvements

⇒ worst-case distribution is a two point distribution

- Wasserstein ball [Pesenti et al., 2020]

Wasserstein uncertainty set

Let $X_0 \sim F_0$ be a **reference distribution** with mean μ and standard deviation $\sigma > 0$.

$$\mathcal{M}_\delta(\mu, \sigma) = \Big\{ X \mid E(X) = \mu, \mathsf{Var}(X) = \sigma^2,\ \hat{d}_W(F_X, F_0)^2 \leq \delta \Big\},$$

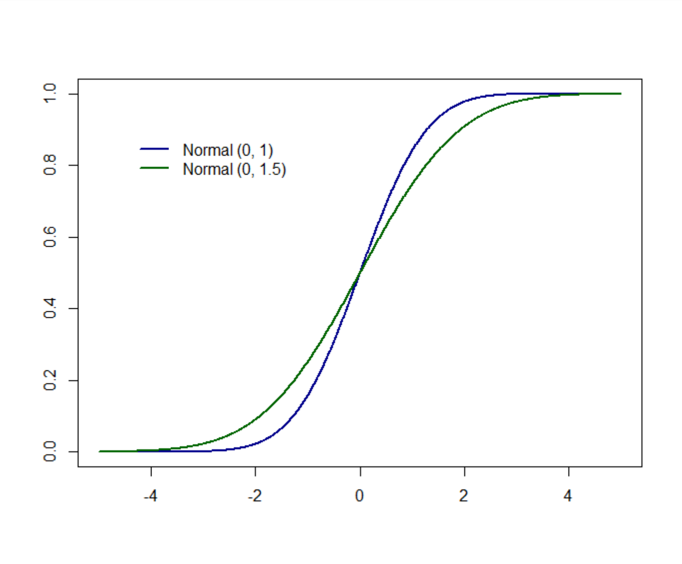
where $\hat{d}_W$ is the "suitably" normalised Wasserstein distance of order 2 such that $0 \leq \delta \leq 1$.

Wasserstein distance

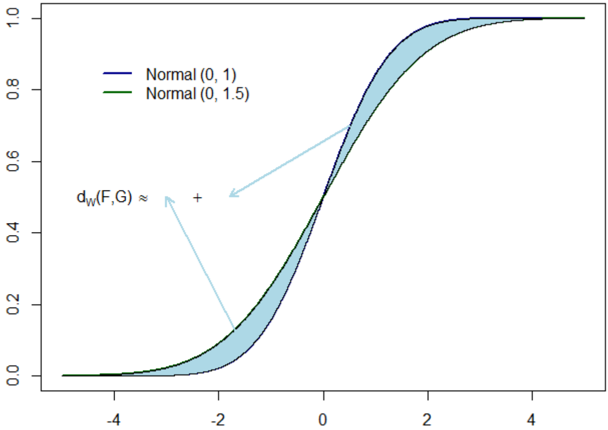
$$
\begin{aligned}
d_W(F,G)^2 &= \int_{\mathbb{R}} \big(F(x) - G(x)\big)^2 \mathrm{d}x, \\
&= \int_0^1 \big(F^{-1}(u) - G^{-1}(u)\big)^2 \mathrm{d}u, \\
&= \inf \Big\{ E\big((X-Y)^2\big) \mid X \sim F,\ Y \sim G \Big\}.
\end{aligned}
$$

Applications: Optimal transport (1781), machine learning, robust statistics, neural networks, Wasserstein Auto-Encoders, image recognition...

Wasserstein distance



Wasserstein distance



Wasserstein bound for ES

Wasserstein bound for ES

For a reference distribution $X_0 \sim F_0$ and tolerance distance $\delta \in [0, 1]$:

$$\left[\inf_{X \in \mathcal{M}_\delta(\mu,\sigma)} \mathsf{ES}_\alpha(X), \qquad \sup_{X \in \mathcal{M}_\delta(\mu,\sigma)} \mathsf{ES}_\alpha(X)\right]$$

with uncertainty set

$$\mathcal{M}_\delta(\mu, \sigma) = \Big\{X \mid E(X) = \mu, \mathsf{Var}(X) = \sigma^2,\ \hat{d}_W(F_X, F_0)^2 \leq \delta\Big\}.$$

Wasserstein bound for ES

$\mathsf{ES}_\alpha(X)$ bounds with reference X_0 and tolerance distance δ:

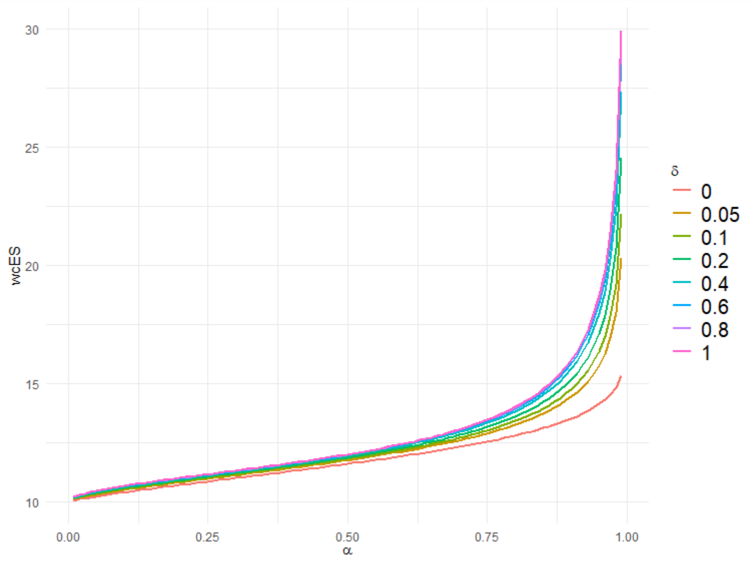
$$\left[\ \mu + \sigma\, c_{\alpha,\lambda}(X_0), \quad \mu + \sigma \frac{\frac{\alpha}{1-\alpha} + \lambda(\mathsf{ES}_\alpha(X_0) - \mu)}{\sqrt{\frac{\alpha}{1-\alpha} + \lambda(\mathsf{ES}_\alpha(X_0) - \mu) + \lambda^2\sigma^2}}\right],$$

where λ is inverse proportional to δ:

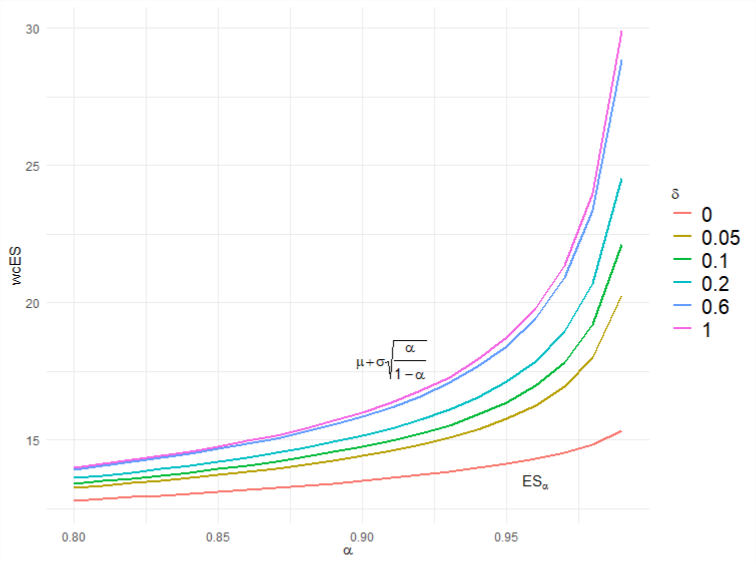
- $\delta = 0$ corresponds to $\lambda = +\infty \quad \rightarrow \quad [\mathsf{ES}_\alpha(X_0), \mathsf{ES}_\alpha(X_0)]$.
- $\delta = 1$ corresponds to $\lambda = 0 \quad \rightarrow \quad \left[\mu, \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$.

Upper bound for ES

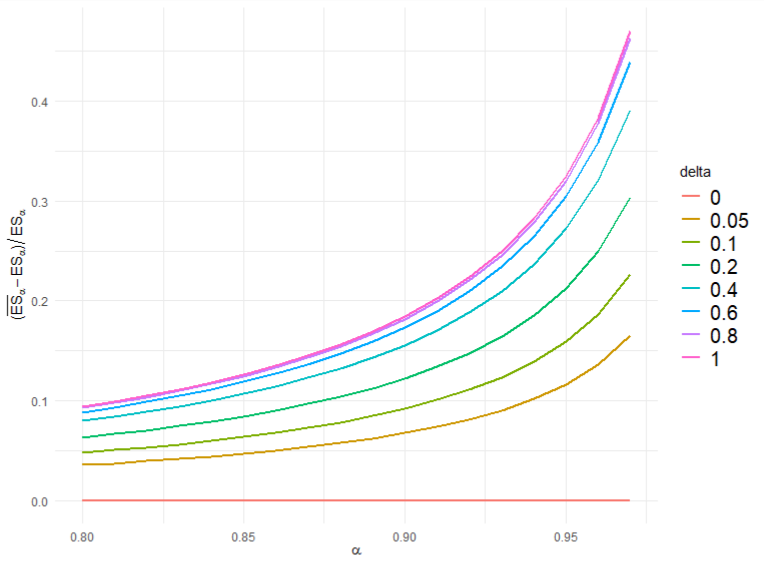
Wasserstein upper bound for ES



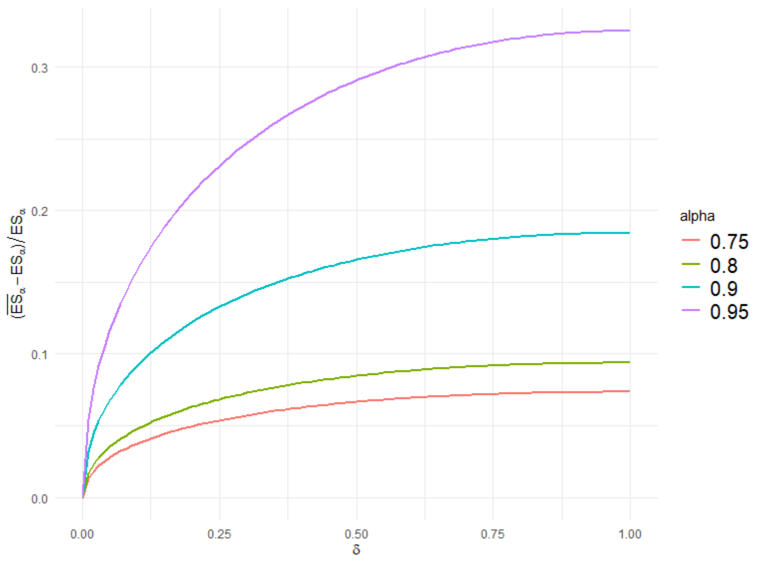
Wasserstein upper bound for ES



Wasserstein upper bound for ES



Wasserstein upper bound for ES



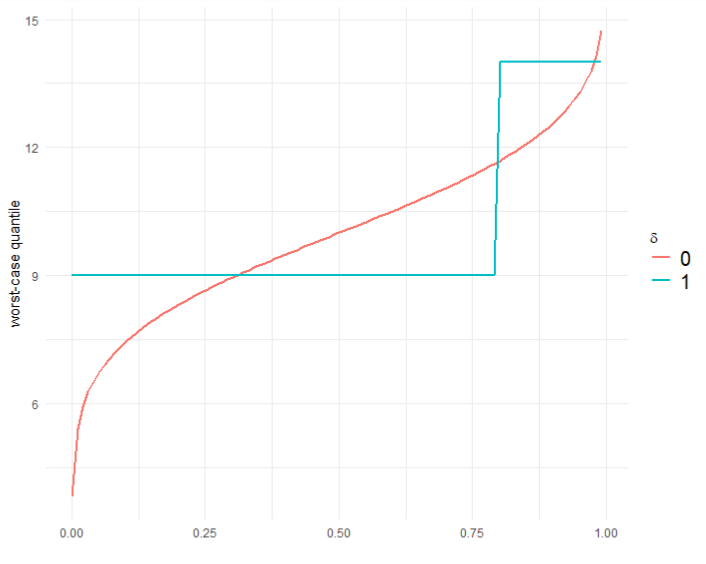
Wasserstein worst-case quantile for ES

The distribution which attains the upper bound, has quantile function

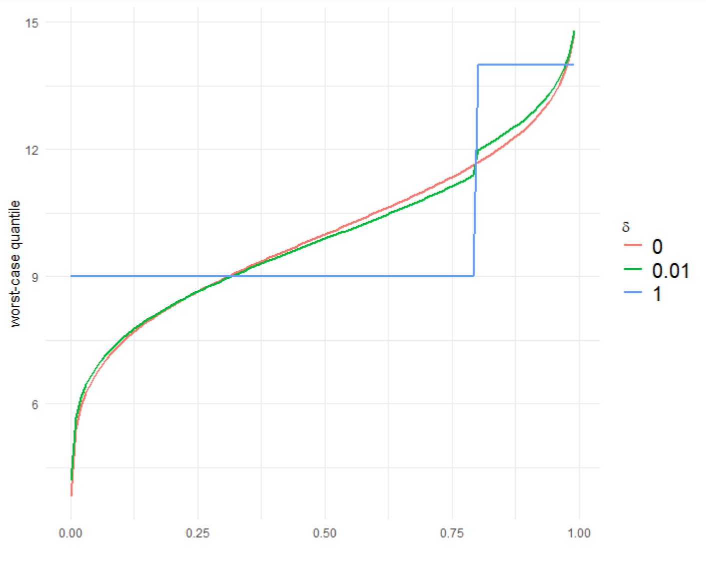
$$F^{-1}(u) = a + b\left(\frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]} + \lambda F_0^{-1}(u)\right),$$

where a, b are such that the mean and standard deviation constraint is fulfilled.

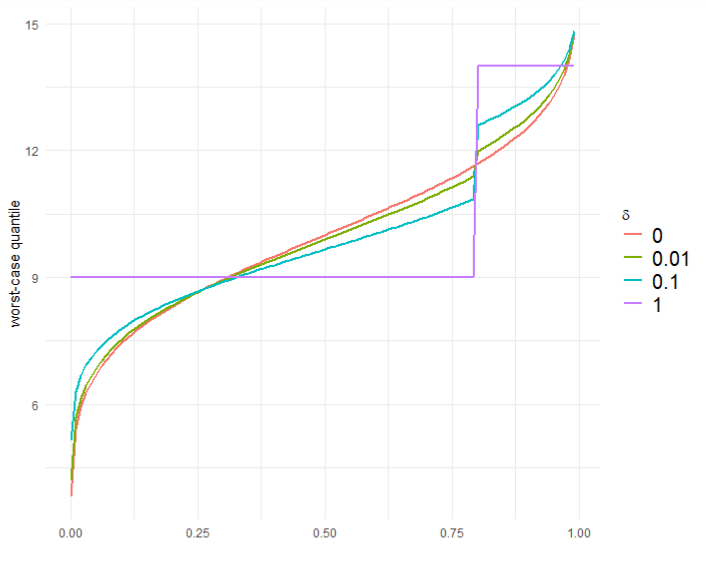
Wasserstein worst-case quantile for ES



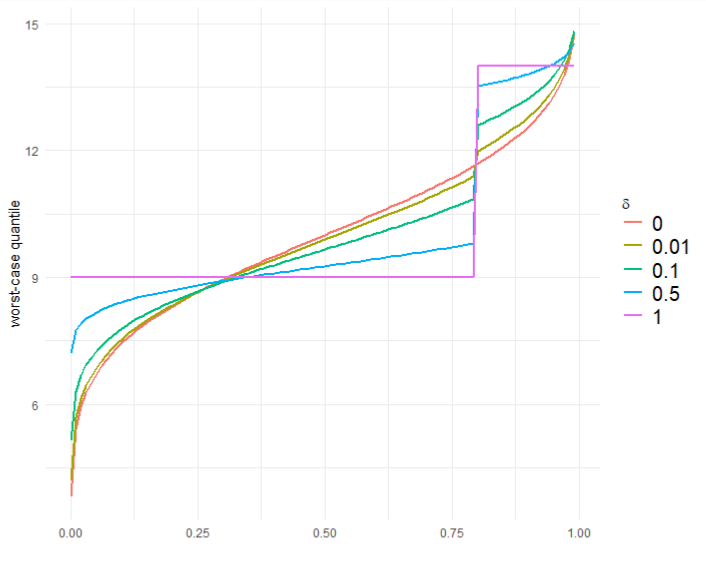
Wasserstein worst-case quantile for ES



Wasserstein worst-case quantile for ES

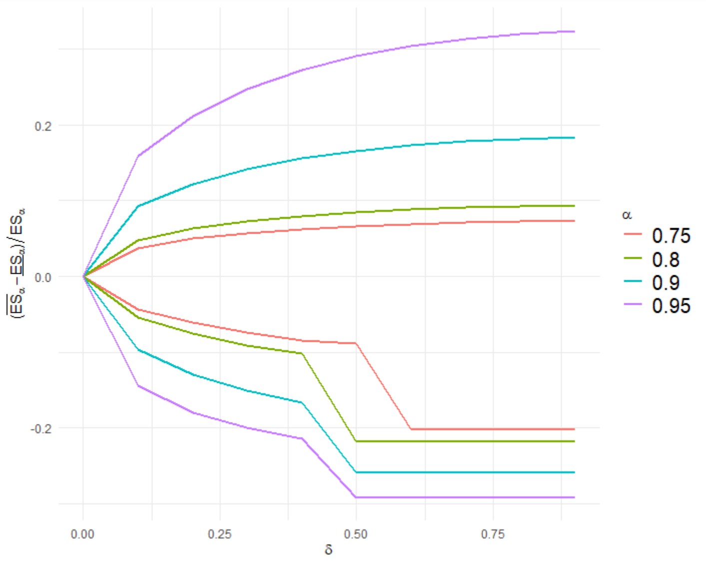


Wasserstein worst-case quantile for ES



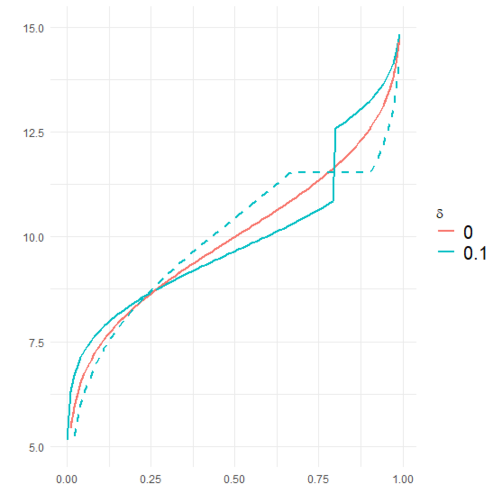
Lower and upper bound for ES

Wasserstein lower bound for ES



Wasserstein best- and worst-case quantiles for ES

The quantile distributions which attain the $\text{ES}_{0.8}$ lower (dashed) and upper (solid) bounds:



Wasserstein bounds in practise

Recipe for deriving Wasserstein bounds

1. Choose reference distribution (*empirical distribution*) with sample mean and sample sd.
2. Choose tolerance distance $\delta \in [0, 1]$.
   $\delta \ll 1$: low uncertainty;
   $\delta \approx 1$: high uncertainty
3. Calculate λ (inverse proportional to δ).
4. Calculate bounds of ES_α.
5. Calculate distribution which attains the bound.

Wasserstein tolerance distance

How to choose the Wasserstein tolerance distance?

a) Distributional uncertainty, expert opinion

b) Model uncertainty, data driven uncertainty set

Wasserstein tolerance distance - distributional uncertainty

a) Distributional uncertainty

Assume we have *uncertainty* in some quantiles:

$$\{\mathsf{VaR}^1_{\alpha_1}, \dots, \mathsf{VaR}^{K_1}_{\alpha_1}, \dots, \mathsf{VaR}^1_{\alpha_M}, \dots, \mathsf{VaR}^{K_M}_{\alpha_M}\}$$

Reason:
parameter uncertainty, expert opinions, additional data sources, ...

Wasserstein tolerance distance - distributional uncertainty

a) Distributional uncertainty

Assume we have *uncertainty* in some quantiles:

$$\{\mathrm{VaR}^{1}_{\alpha_1}, \dots, \mathrm{VaR}^{K_1}_{\alpha_1}, \dots, \mathrm{VaR}^{1}_{\alpha_M}, \dots, \mathrm{VaR}^{K_M}_{\alpha_M}\}$$

Reason:
parameter uncertainty, expert opinions, additional data sources, ...

$\Rightarrow$ Choose δ such that the uncertainty set $\mathcal{M}_\delta(\mu, \sigma)$ is the smallest set containing all quantiles.

Wasserstein tolerance distance - distributional uncertainty

Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

% uncertainty VaR$_{0.8}$, VaR$_{0.9}$	δ	$\underline{\text{ES}_{0.9}}$	$\overline{\text{ES}_{0.9}}$	% bounds
1%	0.013	13.03	14.00	7%
3%	0.030	12.76	14.24	10%
5%	0.061	12.47	14.51	15%
10%	0.209	11.73	15.19	25%

Wasserstein tolerance distance - distributional uncertainty

Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

1% uncertainty in $\text{VaR}_{0.8}$, $\text{VaR}_{0.9}$.

α	δ	$\underline{\text{ES}_\alpha}$	$\overline{\text{ES}_\alpha}$	% bounds
0.9	0.01	13.03	14.00	7%
0.95	0.01	13.36	14.91	11%
0.97	0.01	13.52	15.61	14%
0.975	0.01	13.56	15.87	16%

Wasserstein tolerance distance - model uncertainty

b) Model uncertainty

- F_0 is the *true* unknown distribution.
- Let F_N be the empirical distribution
- Assume the sample mean and sd converge to the mean and sd of F_0

Choose δ such that the true distribution lies in the uncertainty set with probability $1-\beta$. That is

$$P\left(\hat{d}_W(F_N, F) \leq \delta\right) \geq 1-\beta.$$

Wasserstein tolerance distance - model uncertainty

Assume that, for some $\alpha > 2$,

$$E(e^{X^\alpha}) < \infty.$$

Then,

$$\delta \approx \sqrt{\frac{\log(C/\beta)}{N}},$$

where N the sample size and $C \approx 2(E(e^{X^\alpha}) + E(e^{(X/2)^\alpha}) - 1)$.

Wasserstein tolerance distance - model uncertainty

Reference distribution $X_0 \sim \mathcal{N}(10, 2^2)$.

β	N	δ
1%	10^6	0.063
5%	10^6	0.062
10%	10^6	0.061

β	N	δ	% $\text{ES}_{0.9}$ bounds
5%	10^5	0.066	15%
5%	10^6	0.020	8%
5%	10^7	0.007	5%

Wasserstein bounds for VaR

RVaR, VaR

Recall that

$$\mathsf{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathsf{VaR}_u(X) \mathrm{d}u.$$

The methodology for the ES Wasserstein bounds also apply to the RVaR.

RVaR, VaR

Recall that

$$\mathsf{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathsf{VaR}_u(X) \mathrm{d}u.$$

The methodology for the ES Wasserstein bounds also apply to the RVaR.

Moreover, we have

$$\lim_{\alpha' \uparrow \alpha} \mathsf{RVaR}_{\alpha',\alpha} = \mathsf{VaR}_{\alpha}.$$

Thus, we obtain Wasserstein bounds for the VAR.

Recall that

$$\mathsf{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathsf{VaR}_u(X) \mathrm{d}u.$$

The methodology for the ES Wasserstein bounds also apply to the RVaR.

Moreover, we have

$$\lim_{\alpha' \uparrow \alpha} \mathsf{RVaR}_{\alpha',\alpha} = \mathsf{VaR}_{\alpha}.$$

Thus, we obtain Wasserstein bounds for the VAR.
⇒ Future work: assessment of numerical stability of VaR bounds.

Summary

- ▷ Derived bounds for the ES under Wasserstein uncertainty.
- ▷ Wasserstein uncertainty includes distribution with same mean and sd and which are close in the Wasserstein distance.
- ▷ Bounds depend on the reference distribution.
- ▷ Ways of choosing the Wasserstein tolerance distance, via model and distributional uncertainty.

Extension and future work

1. Easily extendable to uncertainty in the mean and standard deviation, e.g. $(\mu, \sigma) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$

Extension and future work

1. Easily extendable to uncertainty in the mean and standard deviation, e.g. $(\mu, \sigma) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$

2. Applicable to any risk measure of the form:

$$\rho(X) = \int_0^1 F_X^{-1}(u)\gamma(u)\mathrm{d}u,$$

for a density γ on $[0, 1]$.

$\Rightarrow$ For example to any *spectral risk measure*.

Future work and discussion

3. Risk bounds for aggregate risks?

$$\inf_{\boldsymbol{X}\in\mathcal{M}} \rho\left(\sum_{i=i}^{d} X_i\right), \quad \sup_{\boldsymbol{X}\in\mathcal{M}} \rho\left(\sum_{i=i}^{d} X_i\right).$$

   a) non-linear aggregation $g(X_1,\ldots,X_n)$?
   b) choice of $\mathcal{M}$?
   c) Incorporating uncertainty in the marginals $X_1,\ldots X_n$?
   d) Incorporating uncertainty in the dependence (copula)?

Thank you!

References

- Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999).
  Coherent measures of risk.
  Mathematical Finance, 9(3), 203–228.
- Cornilly, D., Rüschendorf, L., & Vanduffel, S. (2018).
  Upper bounds for strictly concave distortion risk measures on moment spaces.
  Insurance: Mathematics and Economics, 82, 141–151.
- Embrechts, P., Wang, B., & Wang, R. (2015).
  Aggregation-robustness and model uncertainty of regulatory risk measures.
  Finance and Stochastics, 19(4), 763–790.
- Föllmer, H. & Schied, A. (2011).
  Stochastic finance: an introduction in discrete time.
  Walter de Gruyter.
- Gneiting, T. (2011).
  Making and evaluating point forecasts.
  Journal of the American Statistical Association, 106(494), 746–762.
- Krätschmer, V., Schied, A., & Zähle, H. (2014).
  Comparative and qualitative robustness for law-invariant risk measures.
  Finance and Stochastics, 18(2), 271–295.
- Li, L., Shao, H., Wang, R., & Yang, J. (2018).
  Worst-case range value-at-risk with partial information.
  SIAM Journal on Financial Mathematics, 9(1), 190–218.
- Pesenti, S., Bernard, C., & Vanduffel, S. (2020).
  Robust distortion risk measures.
  Available at SSRN soon.
- Pesenti, S. M., Millossovich, P., & Tsanakas, A. (2016).
  Robustness regions for measures of risk aggregation.
  Dependence Modeling, 4(1).
- Puccetti, G. & Rüschendorf, L. (2012).
  Computation of sharp bounds on the distribution of a function of dependent risks.
  Journal of Computational and Applied Mathematics, 236(7), 1833–1840.
- Wang, B. & Wang, R. (2011).
  The complete mixability and convex minimization problems with monotone marginal densities.
  Journal of Multivariate Analysis, 102(10), 1344–1360.
- Zhu, W. & Shao, H. (2018).
  Closed-form solutions for extreme-case distortion risk measures and applications to robust portfolio management.
  Available at SSRN 3103458.

Equation for λ, upper bound

For $\delta \in [0, 1]$, set

$$\varepsilon = 2\sigma^2\delta \left(1 - \frac{\mathsf{ES}_\alpha(X_0) - \mu}{\sigma\sqrt{\frac{\alpha}{1-\alpha}}}\right)$$

Then, $\lambda \in [0, \infty)$ is the solution to

$$\frac{\varepsilon}{2\sigma^2} = 1 - \frac{\mathsf{ES}_\alpha(X_0) - \mu + \lambda\sigma^2}{\sigma\sqrt{\frac{\alpha}{1-\alpha} + \lambda^2\sigma^2 + 2\lambda\big(\mathsf{ES}_\alpha(X_0) - \mu\big)}}.$$

Wasserstein tolerance distance - distributional uncertainty

The non-normalised Wasserstein tolerance distance is given by

$$\varepsilon = \max_{i=1,\dots,M} \max_{k=1,\dots,K} \int_{\alpha_i}^{1} \left(\mathsf{VaR}^{k}_{\alpha_i} - F_0^{-1}(u)\right)_+^2 \mathrm{d}u.$$